THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
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LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XII., No. 177. JULY, 1925. 2s. 6d. Net.

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The Mathematical Association.

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VOL. XII. JULY, 1925. No. 177.

THE BEGINNINGS OF ARITHMETIC.*

BY PROF. L. N. G. FILON, F.R.S.

1. It is a commonplace of Biological Science that each individual, in his physical development, reproduces in an abridged form the successive stages through which the race has passed in its remote history. Educational experience suggests that a similar process occurs in mental development, so that, generally speaking, ideas have to be introduced in the order of their historical occurrence, if unnecessary resistance is not to be encountered.

A study of the early history of any branch of knowledge is therefore, apart from its interest to the curious, of practical importance to those concerned in teaching it. It is this consideration which has led me to select, as the subject of my remarks this afternoon, the primitive stages in the history of Arithmetic.

2. In tracing the early development of the notion of number we may distinguish three stages.

(i) The *pictorial* stage, in which a number of things of one kind is designated by repeating the picture or symbol of the thing to the required number of times.

Thus we find, in Indian picture-writings, three circles under an arch (Fig. 1, *b*) (Suns under the vault of Heaven) used to denote three days as the duration of an expedition, or a row of fourteen tents (Fig. 1, *a*) to denote the number of lodges, or families, in a tribe. Again, in Cretan inscriptions of the Mycenean age, we find a picture of a ship with two crescents on either side of the mast (Fig. 1, *c*), indicating a voyage lasting two moons.

With numbers above ten this soon gets unendurably cumbrous. The picture is simplified, a single stroke usually taking the place of each object. Sometimes the kind of objects referred to is clear from the context, as when eight upright strokes rising out of a canoe denote a crew of eight rowers (Fig. 1, *d*), or from other considerations, as when a shepherd keeps a record of his sheep by notches on a stick. The Peruvian *quipus* (Fig. 1, *e*), by which tallies were kept by means of knots in differently coloured strings, belong to the same stage.

* Presidential Address to the London Branch of the Mathematical Association, January 31st, 1925.

(ii) The *symbolic* stage, in which the strokes (or some other symbol which replaces them) are placed next to the symbol of the thing to which they apply. By this step the strokes have ceased to represent the object: they have become what is known, in the study of primitive languages, as a *determinative, i.e.* a sign which qualifies the meaning of the object sign. The distinction is fundamentally the same as that between an adjective and a noun: all numerals, in fact, whether cardinal or ordinal, are really adjectival; *five men* really implies men (five in number).

This step definitely isolates number as an abstract quality.

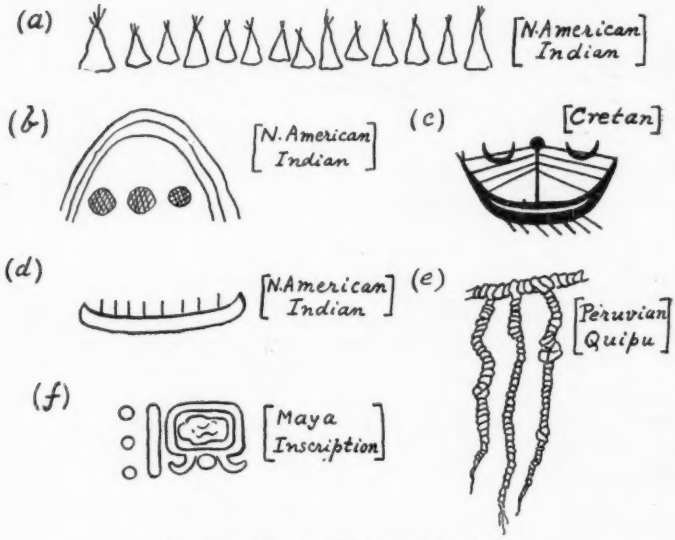


FIG. 1.—Evolution of Numerals from Picture-writing.

(iii) The *calculation* stage arises out of the symbolic, namely, when rules are evolved for combining the numerical symbols and arriving at new results, independently of the kind of object to which they are applied. When this stage is reached, arithmetic proper may be said to have begun.

3. So far as historical records go, we find that in every case, in the ancient civilisations known to us, the calculation stage has been reached, but with very varying success, according to the capabilities of the symbolic notation employed. In fact, it is not too much to say that the whole development of arithmetic has been largely a question of notation, and in what follows it is to that side of the matter that I propose to give most attention.

It is clear that the method of adding single strokes soon comes to an end as the numbers increase, for the number of objects which the eye unhesitatingly recognises immediately is very limited. Beyond four or five, we have to depend on *counting*.

But counting by single units involves reference to a standard sequence; ultimately this standard sequence must be either a set of material objects, *e.g.* the fingers, or the beads of an abacus, or the names of the natural numbers; but again, the latter soon come to a stop without a system of numeration, as

is easy to realise if one but thinks of having to count up to a hundred through a set of individual names not systematically related.

In practice, therefore, counting necessarily takes the form of counting in groups, and then counting the number of groups; then forming groups of groups, and so on.

The number of the fundamental group is what we call nowadays the *base* of the system of numeration. With the exception of the sexagesimal system, which arose at a later stage of development and for very much more artificial reasons, all the primitive systems known to us take five, ten, or twenty as the fundamental groups. These correspond to the numbers which can be counted on the fingers of one or two hands, supplemented it may be by the toes. We often find two fundamental numbers, *e.g.* 5 and 10, used; but 10, at any rate in the arithmetic of the old world, is always dominant.

4. A system of numerical notation may be of one of two kinds.

(*a*) There may be a distinct symbol for every kind of group, and these may be repeated as often as necessary.

A simple and familiar example is the original form of the ordinary Roman numerals.

The *unit* is denoted by a single upward stroke I (*digitus*, a finger).

Five is denoted by V, which represents the hand, four fingers close together and thumb extended.

Ten is denoted by X, which represents two hands.

These are clearly very ancient symbols, derived from picture-writing. Later symbols are ordinary letters of the Latin alphabet: L for 50, C for 100, D for 500, M for 1000.

In the original form of this system, every number may be expressed (up to 5000) by a set in which the same symbol need never be repeated more than four times.

Thus $2174 = \text{MMCLXXIIII}$.

Although in writing the above the various groups have, according to the usual custom, been arranged in *descending* order of magnitude, it is clear that the value of the number has nothing to do with their position, and the number is simply the sum of the groups denoted by the various symbols.

Such a system I will call for shortness an *additive* notation.

A pure additive notation is by no means without its advantages for purposes of calculation; with it the process of addition is almost instinctive, and that of multiplication, though slow, involves very little strain, since only multiples of 5 and 10 need be remembered. This will be clear from the following examples:

(i) Addition:

$$\begin{array}{rl} 517 = & \text{DXVII} \\ 1239 = & \text{MCCXXXVIIII} \\ \hline 1756 & \text{MDCCLVI} \end{array}$$

since $\text{IIIII} = \text{V}$, $\text{VV} = \text{X}$ and $\text{XXXXX} = \text{L}$.

(ii) Multiplication: $53 \times 37 = 1961$

$$\begin{array}{rl} & \text{XXXVII} \\ & \text{LIII} \\ \hline \text{XXXVII} \times \text{I} = & \text{XXXVII} \\ \text{,,} \quad \times \text{,,} = & \text{XXXVII} \\ \text{,,} \quad \times \text{,,} = & \text{XXXVII} \\ \text{,,} \quad \times \text{L} = & \text{MDCCCL} \\ \hline & \text{MDCCCCLXI} = 1961 \end{array}$$

since $\text{L} \times \text{X} = \text{D}$, $\text{L} \times \text{V} = \text{CCL}$, $\text{L} \times \text{II} = \text{C}$, and $\text{DDD} = \text{MD}$.

The defect of the additive notation is that it is usually exceedingly cumbrous as soon as the numbers are at all large. In the case of the Roman notation,

as we know it, it has been entirely spoilt for calculation by the modification whereby when a symbol precedes the next higher one, it is to be subtracted, thus:

IV = 4
IX = 9
41 = XLI
1925 = MCMXXV
449 = CDXLIX

(*b*) A symbol may have a value in virtue of its *position*; the obvious example is our present system, where a digit may stand for a number of units or the same number of tens or of hundreds, according to where written. A system of this kind I shall term a *positional* system. Here the groups are denoted entirely by position, and a limited number of symbols are required to give the number of each group.

The essence of the system is that position shall be unmistakably defined, and this involves that a symbol shall be used to denote the absence of a particular type of group—or, what comes to the same thing, that a blank for that type of group shall be left. The rules of calculation are more elaborate, and a multiplication table is necessary. On the other hand, the system is capable of indefinite extension to numbers as great as we please.

5. The earlier notations of the civilisations of antiquity (Egyptian, Babylonian, Greek and Roman) are all additive, and the history of arithmetic is largely that of the evolution from the additive to the positional notation.

	Egyptian.	Chaldean.	Attic.	Roman.
1				I
5				V
10				X
50				L
100				C
500				D
1000				M
5000				
10,000				
50,000				
100,000				
1,000,000				

Fig. 2.—The Primitive Additive Systems. Simple Symbols.

A kind of intermediate stage has been discovered in America in the partially-deciphered inscriptions of the Mayas of Yucatan, whose civilisation flourished in the first millennium of the Christian Era. Their numeral notation went no further than 20, a dot representing a unit, and a bar five units.

Thus $\vdots|||=17$, $\vdots|=9$.

The Maya inscriptions refer to dates, and these are expressed in lustres (Katun) of 20 years (Tun); 20 lustres made a cycle, 20 cycles a great cycle. The year was divided into 18 months of 20 days, with 5 supplementary days. The cycle, lustre, year, etc., each had its own symbol, to which the number was prefixed (see Fig. 1, *f*). Thus the twenty numerical symbols were adequate to express any date.

The system is interesting as being an example of a *vigesimal* system, *i.e.* one based on twenty. None of the European or Near Eastern ancient systems gives prominence to the number *twenty*, although evidence that such a base must have been in use in Western Europe at one time is afforded by the survival of such terms as "Three score and ten" in English and "Quatre-vingt" in French.

6. The earliest numerical notations known to us are the Egyptian and Chaldean, or Babylonian.

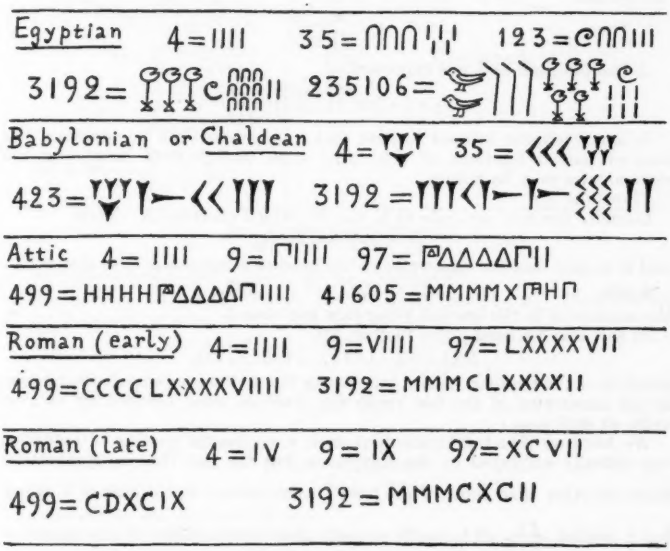
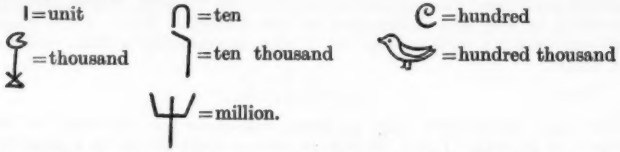


FIG. 3.—The Primitive Additive Systems. Methods of writing Numbers.

The Egyptian notation is definitely based on *ten*.

The chief source of our information about this is the so-called *Ahmes papyrus*, which is a kind of compendium of arithmetic, and is supposed to date from about 1700-2000 B.C.

The signs are as follows:



The method of writing the number was the same as in the Roman systems, except that—there being no special symbols for the multiples of 5—lateral space was saved by arranging the figures in two or three rows, one above the other.

The numbers could be written either from left to right or from right to left; in the latter case the signs were turned the opposite way.

Thus $4197 =$

There existed also an Egyptian notation for fractions, but the only fractions considered were those having numerator unity, and consisting of single aliquot parts. Such a fraction was denoted by writing the denominator below the symbol .

Thus $\frac{1}{213} =$

A fraction such as $\frac{7}{8}$ was expressed as

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8} =$

It is not without interest to note that *every fraction* can be expressed as a sum of partial fractions of this form, even though both numerator and denominator may be primes.

Thus take $\frac{5}{13}$.

Consider the first multiple of 5, viz. 15, which exceeds 13. Then

$$\tfrac{5}{13} = \tfrac{1}{3}(\tfrac{15}{13}) = \tfrac{1}{3}(1+\tfrac{2}{13}),$$

and it is clear that the numerator in the bracket is necessarily less than 5.

Again, $$2/13 = (\tfrac{14}{13})/7 = (1+1/13)/7,$$

the numerator in the bracket being now less than 2.

In this case we need go no further, for

$$5/13 = 1/3 + 1/(3.7) + 1/(3.7.13),$$

which is of the required form; but clearly the method is general, the number in the numerator of the last remaining fraction being reduced by at least unity at each step.

We have no direct evidence that such a systematic process of reduction was actually employed by the Egyptians, but the fact that no symbols for fractions other than those of the type $\frac{1}{n}$ exist (except in the case of $\frac{2}{3}$, which had a symbol all to itself) suggests that the possibility of this reduction in every case must have been at least suspected by them. If this is so, it implies a very considerable knowledge of number theory at this early date.

7. The methods of addition and subtraction, on this system, are obviously the same as those explained for the Roman system, and lend themselves easily to the use of sliding beads in an abacus, a method of computing used by the Japanese to this day.

Their methods of multiplication and division, however, are of extraordinary interest, and they suggest a stage in which *two* was the basis of numeration.

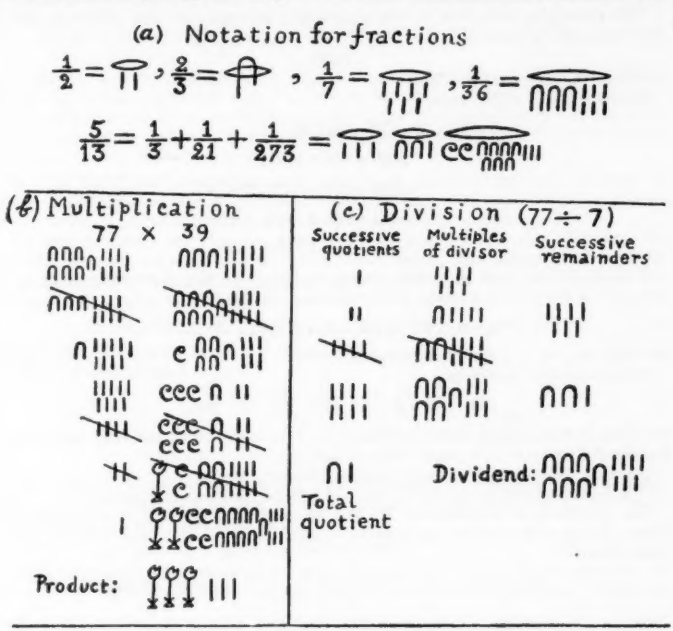


FIG. 4.—Egyptian Arithmetic.

This will be made clear by the following examples:

Suppose they desired to multiply, say, 77 by 39 (see Fig. 4). They divided 77 by 2, leaving a remainder 1, the quotient being 38.

Then $77 \times 39 = (1 + \{2 \times 38\})39 = 39 + \{38 \times 2\}\,39$
$= 39 + (38 \times 78).$

They wrote down

77	39
38	78

In this case dividing 38 by 2, and multiplying 78 by 2, we have

$$38 \times 78 = 19 \times 156$$

19	156

The number 78 was then struck out.

On again dividing by 2, we obtain

$$19 \times 156 = 156 + \{9 \times 2(156)\} = 156 + \{9 \times 312\}$$

	9	312	$9 \times 312 = 312 + 4(624)$
Repeat	4	~~624~~	$4(624) = 2(1248)$
	2	~~1248~~	$= 1.(2496)$
	1	2496	

The numbers in the right-hand column corresponding to *even* numbers in the left-hand column being struck out.

The remaining numbers in the right-hand column are then *added together*; their sum will be found to be:

3003,

which is the required product.

This is clearly right, since

$$\begin{aligned} 77 \times 39 &= 39 + 19(156) \\ &= 39 + 156 + 9(312) \\ &= 39 + 156 + 312 + 4(624) \\ &= 39 + 156 + 312 + 2(1248) \\ &= 39 + 156 + 312 + 2496. \end{aligned}$$

Now it will be seen that this system amounts to the expressing of 77 as a number on the *binary* system of numeration, in the form

$$a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 + a_4 \cdot 2^4 + \dots,$$

the digits a_0, $a_1 \dots$, etc., being either 0 or 1.

The successive products

$$39, \quad 39 \times 2, \quad 39 \times 2^2, \quad 39 \times 2^3, \text{ etc.},$$

are then formed, those corresponding to a zero digit being omitted, and those corresponding to a unit digit being added up.

Division was worked out on precisely similar lines.

If it was required to divide 77 by 7, the procedure was as follows:

The products of the divisor by the successive powers of 2 were written down, being obtained by successive doubling, until one was reached just lower than 77.

Power of 2.	Product by 7.	Successive remainders.
1	7	0
2	14	7
~~4~~	~~28~~	
8	56	21
11		

This is 56, and $77 - 56 = 21$. The next lower product in the sequence below 21 is then taken (*i.e.* 14), the intermediate one (28) being struck out. $21 - 14 = 7$, and the next lower product is 7, which leaves no remainder.

The powers of 2 not struck out are then added up, giving 11, which is the required quotient.

Smith Williams, in his *History of Science*, describes this method, but appears to think it one of trial and error; this is wrong, the method is perfectly systematic, and, in fact, involves less trial and error than our own method of long division.

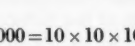
It exhibits the quotient as a number

$$(1 \times 2^3) + (0 \times 2^2) + (1 \times 2) + 1$$

on the binary scale.

8. Coming now to the Babylonians, we find two systems in use. Probably the earlier and more fundamental one is the decimal notation. This has three fundamental symbols, in the usual wedge-shaped characters of the cuneiform inscriptions:

Higher powers of ten were expressed by combinations of these symbols, on a multiplicative basis. Thus

$1000 = 10 \times 100 =$  ; $10{,}000 = 10 \times 10 \times 100 =$,

and these groups were apparently regarded as single signs (see Fig. 2). These single signs were repeated, much as in the Egyptian system, but we are still largely ignorant of the rules which governed the interpretation of the signs for higher numbers, and which determined when a symbol was to have multiplicative value, and when additive value.

9. The Babylonian sexagesimal system is of profound interest, because it is the first instance of a numerical symbol having *positional* value, a notion which is the basis of our present system of numerical notation.

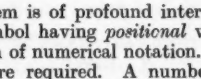
In this system, only 60 numbers are required. A number between 60 and 60^2 is expressed as so many $60 +$ so many units; one between 60^2 and 60^3 as so many 60^2, so many 60 and so many units. In fact, any number N, however large, can be written:

$$N = a_n(60)^n + a_{n-1}(60)^{n-1} + a_{n-2}(60)^{n-2} + \ldots + a_1(60) + a_0,$$

where the numbers a are all less than 60.

Thus, in Babylonian script,

$$98{,}496 = 27 \,.\, 60^2 + 21 \,.\, 60 + 36$$

$=$ 

Not merely so, but the system was extended to sexagesimal fractions, which were worked out to any required degree of approximation, so that a proper fraction was expressed as

$$b_1/60 + b_2/60^2 + b_3/60^3 + \ldots .$$

Such a system only required a symbol for *zero* and a fixed arrangement of columns to become in every way as good as our present notation. There is some evidence that a sign was occasionally used, to indicate that a group or class was absent, or else a gap was left.

It involved, however, a formidable multiplication table, extending to 59×59.

10. The origin of this sexagesimal system is not definitely known, but one may hazard the guess that it was probably astronomical and connected with their division of the circle, which has persisted to the present day. There is no doubt that all primitive calendars are based on the belief that the year is 360 days and the lunar month 30 days. The *degree*, on this interpretation, is the distance travelled by the sun among the stars in a single day. There is an interesting piece of collateral evidence bearing on this, which is highly suggestive: it is known that the Chinese circle was divided into $365\frac{1}{4}$ degrees; this makes the degree correspond to the correct length of the year, and shows that the Chinese, as their observations improved, modified their circular division to suit them. The Babylonians must have known from quite early times—probably from 3000 or 4000 B.C.—that the length of the year was not 360 days, but they nevertheless retained this convenient division of the circle.

Now the angle of 60°, namely the angle of the equilateral triangle, is an angle the construction of which is obvious: in some ways more obvious than that of the right-angled triangle. Moreover, it is the angle whose chord is equal to the radius, and, as the length of a chord was largely used by the earlier geometers for constructing angles, the angle of 60° naturally suggests itself as the unit angle. The degree being 1/60th part of this, a division into sixtieths

is indicated, the sixtieth of a degree being a minute, and the sixtieth of a minute being one second: a division which is still in use, so that we may say the sexagesimal system of the Babylonians has persisted to this day.

11. Coming now to the Greeks, we find two entirely distinct notations. The first is known as the Attic notation; it is found in Athenian inscriptions from 454 B.C. to about 95 B.C. It is very similar to the Roman notation, and is probably the primitive notation of the Greeks.

Except for the unit (I) the symbols for 5, 10, 100, 1000, 10,000 are the initial letters of the corresponding names; thus

$\Pi = 5$ (πέντε); $\Delta = 10$ (δέκα); $\mathrm{H} = 100$ (ἑκατον).

(H is our "h," which was replaced by the "rough breathing" in classical Greek.)

$\mathrm{X} = 1000$ (χίλιοι); $\mathrm{M} = 10{,}000$ (μύριοι).

50, 500, 5000, 50,000 were denoted by inscribing the corresponding symbols in Π, thus $\overline{\Delta|}$, $\overline{\mathrm{H}|}$, $\overline{\mathrm{X}|}$, $\overline{\mathrm{M}|}$. (Cf. Fig. 2.)

Numbers are then formed precisely as in the primitive Roman notation, no subtractive terms occurring, so that the system is purely additive, although, as a matter of fact, the larger numbers are written before the smaller ones, thus:

$4999 = \mathrm{XXXX}\overline{\mathrm{H}|}\mathrm{HHHH}\overline{\Delta|}\Delta\Delta\Delta\Delta\Pi\mathrm{IIII}.$

12. The second Greek notation, which eventually replaced the other, but disappeared from Europe during the Dark Ages, being replaced by the more backward Roman system, is that of the so-called alphabetic numerals.

In this notation the numbers of units, tens, and hundreds, which could occur in any number from 1 to 999, were denoted by the letters of the Greek alphabet, supplemented by certain letters which are obsolete in classical Greek. The arrangement is as follows:

	Units			Tens			Hundreds	
1	Α, α	Alpha	10	Ι, ι	Iota	100	Ρ, ρ	Ro
2	Β, β	Beta	20	Κ, κ	Kappa	200	Σ, σ	Sigma
3	Γ, γ	Gamma	30	Λ, λ	Lambda	300	Τ, τ	Tau
4	Δ, δ	Delta	40	Μ, μ	Mu	400	Υ, υ	Upsilon
5	Ε, ε	Epsilon	50	Ν, ν	Nu	500	Φ, φ	Phi
6	(Ϝ), C, ς	(Digamma) Vau, Stigma	60	Ξ, ξ	Xi	600	Χ, χ	Chi
7	Ζ, ζ	Zeta	70	Ο, ο	Omicron	700	Ψ, ψ	Psi
8	(Θ), Η, η	(Cheth) Eta	80	Π, π	Pi	800	Ω, ω	Omega
9	Θ, θ, ϑ	Theta	90	Ϙ	Koppa	900	Ϡ, ϡ	Ssade

FIG. 5.—Greek Alphabetic Numerals.

These alphabetic numerals originated probably in Greek Asia Minor in the fifth or sixth centuries before Christ. They were not officially used in Greece proper until the time of the Ptolemies.

With this system a number was represented by precisely the same number of figures as with us, the number being usually read from left to right (in European Greece), thus:

$$976 = \overline{\text{ϡοϛ}}.$$

FRACTIONS

$1/3 = \gamma'$ $1/32 = \lambda\beta'$; special signs: $1/2 = \iota'$ or $\subset'$; $2/3 = \omega'$

$365\frac{31}{128} = \overline{\tau\xi\epsilon}\ \overline{\lambda\alpha}\ \rho\kappa\eta^{\omega\nu} = \overline{\tau\xi\epsilon}\ \lambda\epsilon\pi\tau\acute{\alpha}\ \rho\kappa\eta'\rho\kappa\eta'\overline{\lambda\alpha}$ (Heron)

$= \overline{\tau\xi\epsilon}\ \overline{\lambda\alpha}^{\rho\kappa\eta}$ (Diophantus)

SEXAGESIMAL NOTATION

$47^\circ 42' 40'' = \mu o\iota\rho\tilde{\omega}\nu\ \mu\zeta\ \mu\beta'\ \mu''$

$0^\circ\ 1'\ 2''\ 0''' = \mu^\circ\ 0\ \alpha'\ \beta''\ 0'''$

chord $120^\circ = \sqrt{3} = \tau\mu\eta\mu\acute{\alpha}\tau\omega\nu\ \rho\gamma\ \nu\epsilon'\ \kappa\gamma'' = \frac{103}{60} + \frac{55}{(60)^2} + \frac{23}{(60)^3}$

$= 1{\cdot}7320509$ (True value $= 1{\cdot}7320508$)

FIG. 6.—Numbers in Greek Alphabetic Notation.

The numerals were distinguished from the ordinary text by having spaces on either side, or enclosing the number between dots, ⁝κδ⁝, or, more generally, in Imperial times, by writing a bar over the symbols.

The system was extended to higher numbers in the following way: Thousands were written by taking the letters corresponding to the corresponding units and writing an accent to the left, thus:

$$,\zeta \text{ or } '\zeta = 7000.$$

For tens of thousands the letter M (*μυριάδες*) was written under, or above, or next to the number. This carried the system as far as 10^8, which sufficed for all practical purposes. To express still higher numbers, second myriads $(10{,}000)^2 = \overset{\text{B}}{\text{M}}$, third myriads, $\overset{\Gamma}{\text{M}}(=10{,}000)^3$; etc., could be introduced. Such symbols are found in the works of the mathematicians Diophantus, Apollonius and Pappus.

Examples of large numbers written in these various ways are:

$$\overset{,\xi\rho o\epsilon}{\text{M}}\ ,\epsilon\omega o\epsilon = 71755875 \text{ (Aristarchus of Samos).}$$

$$\overset{\Gamma}{\mu},\epsilon\upsilon\xi\beta\ \kappa\alpha\iota\ \overset{\beta}{\mu},\gamma\chi\ \kappa\alpha\iota\ \overset{\alpha}{\mu},\varsigma\upsilon = 5462\ 3600\ 6400\ 0000 \text{ (Pappus).}$$

Note here the faint anticipation of our notation for the *power* and exponent.

This Greek alphabetic notation was therefore partly on the old plan of giving selected numbers a symbol, and partly positional. Here, again, what was lacking was the zero, and the notion of repeating the digits. It was spoilt by the use of separate symbols for the tens and the hundreds. It involved a greater strain on the memory, when small numbers were involved, than the Roman or Attic numerals, but it was very much more compact and could deal with much larger numbers.

13. Side by side with this decimal system, and using the same numerals, there existed a sexagesimal system, following much the same lines as the Chaldean, which was used for calculating square roots, etc., in the same

way in which we now use decimal fractions. The system is found in its complete form in Ptolemy's *Almagest*, in which it refers not only to minutes and seconds of arc, but also to subdivisions of unit length, the chord of an angle of 60° being itself divided into 60 equal parts, the parts into first sixtieths, these again into second sixtieths, and so on.

The degrees or parts were called μοῖραι or τμημάτα and denoted by μ° (hence our ° for degree, μ having disappeared). The first sixtieths were denoted by an accent ′, the second sixtieths by two accents ″, and so on. For example,

$$\mu^\circ\ \mu\zeta'\ \mu\beta''\ \mu'''=47^\circ\ 42'\ 40''.$$

EUROPEAN		ARABIC	POSSIBLE TRANSITION	GREEK	INDIAN			HYPOTHETICAL STROKE SYMBOLS
MODERN	XIIth CENTY				Xth CENTY	Vth CENTY	1st CENTY	
1	1	1	1,1	A,α	1	1	—	1
2	2	2	2	B,β	2	2	=	2
3	3	3		Γ,γ	3	3	≡	3
4	4	٤	4	(Δ),Δ,δ	4	4	4	4
5	5	5	5	E,ε	5	5	5	5
6	6	٦	6	(F),Ϛ,ς	6	6	6	6
7(7)	7	7	7	Z,ζ	7		7	7
8	8	٨	8,8	(Θ),H,η	8		8,3	8
9	9	9	9	Θ,θ,ϑ	9		9	9
0	0		0	O,o	0,·	0		

FIG. 7.—Derivation of Arabic Numerals.

But this system was remarkable in that it definitely used for the first time a *zero*. The symbol O was used when a particular unit or subunit was not represented. Two explanations of the origin of this O have been given. The first is that, since the actual figures written down in the sexagesimal system could never exceed 60, the next symbol in the list (o=70) was used to denote a blank. The second is that O is the initial letter of the Greek word οὐδέν (none), and, having regard to the similar use made by the Greeks of initials in other cases, it seems the more likely explanation. Be that as it may, it seems most probable that we have here the origin of our zero.

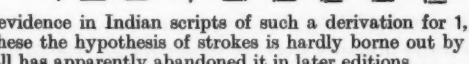
The methods of calculation, with this system, can be gathered from examples in the writings of Greek mathematicians. Multiplication and division pro-

ceeded on lines very similar to ours; the multiplicand being multiplied in succession by the various terms of the multiplier and the results added; division was carried out by repeated subtraction.

There was also a method of obtaining square roots to a given degree of approximation, in connection with the sexagesimal system, and Ptolemy actually works out $\sqrt{3}$ correct to one in the seventh decimal place.

14. The origin of the Arabic numerals, which are now universally used, is still obscure. It used to be thought that they were invented in India, then adopted by the Arabs, and that through the Arabs they found their way into Spain and, at the time of the Crusades, into the Western world.

Rouse Ball, in the first edition of his *History of Mathematics*, suggests a derivation from a set of symbols made up of the number of strokes required to give the number, the ends of the strokes being joined up in cursive writing, thus:



There is evidence in Indian scripts of such a derivation for 1, 2 and 3, but beyond these the hypothesis of strokes is hardly borne out by the facts, and Rouse Ball has apparently abandoned it in later editions.

Moreover, recent research has thrown some doubt upon the antiquity of Hindu mathematics. The evidence on which we largely depend in this connection is a Hindu treatise on Astronomy, called the *Sūrya Siddhānta*, which was probably composed in about A.D. 500, and which seems to have derived a great deal from the Greek Alexandrian School.

There is no necessity, however, to suppose that the Arabic numerals were derived from the Greeks through the Hindus. They may well have arisen in the Near East itself. The Greek alphabetic numerals show signs of Phoenician or Semitic origin. In particular they retain Hebrew letters, such as *Vau*, *Qoph*, *Ssade*, which have disappeared in classical Greek. Moreover, the Hebrews possessed a system of numeration practically identical in principle with the Greek alphabetic numeral system. Also *Vau* and *Qoph*, though not *Ssade*, occur in the Greek numerals in the same place which they have in the Hebrew alphabet. All this points to the Greek alphabetic system having been derived from a Semitic source; it would therefore be only natural that it should have persisted and been improved in its country of origin.

After the collapse of the Roman Empire, the Arabs inherited the scientific tradition of Alexandria, and there is no doubt that they must, through the writings of Ptolemy and others, have been made familiar with the Greek numeral symbols.

Now it is very remarkable that these symbols (at any rate as regards 6, 7, 8 and 9) bear a singular resemblance to the corresponding Greek letters.

Thus, the late forms of *Vau* (capital and cursive) are and the latter is almost identical with the sign for 6 in Indian scripts, and it is easy to see how it might easily become 6.

Zeta, Z, ζ, is almost identical with an archaic form of 7. Moreover, even now, on the Continent, the 7 is often *crossed*, thus .

8 is specially interesting. The Greek *Eta* is the Hebrew *Cheth*, and is originally , which would lead almost inevitably to 8 as a cursive form.

Theta, , θ, would lead easily to 9 (which in fact is one of its cursive forms), and from this to 9 is but a step.

How E, ϵ, would lead to 5 is not at first obvious; but we find for 5 the form in Arabic manuscripts and in European twelfth-century MSS., and the relation to ϵ is here evident.

4 is a serious difficulty. It is true that is an old form of Δ, which leads at once almost to 4, but, unfortunately, the intermediate forms do not bear this out. An early Indian symbol is , which does look like a "four stroke" symbol: this becomes, in later Indian script, ; the Arabic symbol is ; the European twelfth-century symbol is . It is possible that the Arabic symbol is derived from the cursive , through the transformation and turning the symbol round—a frequent process.

3 appears to derive definitely from the three strokes. The Arabic symbol is , and the early European .

Similarly, 2 may well derive from the two strokes. The Arabic symbol is .

The single stroke for the unit is a symbol hallowed by tradition, and would naturally have replaced any form of α.

But the decisive argument is really supplied by the zero. This is absent from the earlier Indian scripts, or else zero is denoted by a *dot*. On the other hand, we have definite evidence that 0 is used to denote zero in Ptolemy's *Almagest*, which the Arabs had thoroughly mastered.

If this argument is correct, the Arabic numerals have really been derived from the Greek alphabetic numerals, by omitting the separate signs for tens and hundreds, and by importing three new signs for 1, 2 and 3.

But the value of each sign was indicated by its position. That, and the systematic use of the zero, made our arithmetic possible. L. N. G. FILON.

GLEANINGS FAR AND NEAR.

317. The Stirling Jug.—The Stirling jug, now in the Museum at Stirling, is a brass jug, which was made to the standard pint measure for Scotland by the order of David II. in 1457. People wishing to get their measures tested had to come to Stirling for that purpose.

About 1750 the Rev. Mr. Bryce, of Kirknewton, came to Stirling, and wished to see this standard jug. He was shown a pewter one, but he would not accept that as the right one. He searched for two years, and at last found the real jug in a pewterer's attic. For some reason the pewterer had got the loan of the brass jug, but having joined up at the 'Forty-five rebellion had not come back to his business in Stirling.

Mr. Bryce restored the jug to Stirling. For that and other acts the freedom of Edinburgh was conferred upon him, but Stirling returned no thanks for his trouble.—C. TWEEDIE.

318. *Practical Arithmetic for the Use of Adults.* By Sergeant Fletcher, Scots Fusilier Guards.—Sergeant Fletcher is a credit to the regiment; and we have no doubt is fit to teach all the officers. There is thought about his book.—*Athenaeum*, Aug. 12, 1848.

HOW TO SOLVE DIFFERENTIAL EQUATIONS APPROXIMATELY BY ARITHMETIC.

By Lewis F. Richardson, F. Inst. P.

I. *The first obstacle may be one of sentiment.* It is said that in a certain grassy part of the world a man will walk a mile to catch a horse, whereon to ride a quarter of a mile to pay an afternoon call. Similarly, it is not quite respectable to arrive at a mathematical destination, under the gaze of a learned society, at the mere footpace of arithmetic. Even at the expense of considerable time and effort, one should be mounted on the swift steed of symbolic analysis.

The following notes are written for those who desire to arrive by the easiest route, and who are not self-conscious about the respectability of their means of locomotion.

The method of solution by arithmetic is so different from that by analysis that either method may be cumbrous in a problem where the other is easy.

The distinguishing characteristic of the methods now to be described is that the passage towards the limit $\delta x \to 0$ is made at the end of the calculation, instead of before it as in analysis, or during it, as in many better-known arithmetical methods.*

This postponement of the approach to the limit allows all the previous operations to be very simple, and very adaptable to divers problems.

We deal in real variables, imaginaries being excluded.

II. *The number of variables, the order and degree of the equation,* all of which would have such a profound influence on the method of solution, if it were to be analytical, have, when arithmetic is employed, a much diminished rôle, so that we need not pause to consider them until the distinction between "marching" and "jury" problems has been made out, as in VI. below.

III. *Consider the given differential equation, not alone, but in conjunction with any boundary—or other—conditions which make the problem definite,* for all that arithmetic can do is to give us "particular integrals." For example:

Ex. (i). $\sigma \partial^2\phi/\partial x^2 = \partial\phi/\partial t$, where σ is a constant, might be made definite by having given $\phi = 1$ when $t = 0$; $\phi = 0$ when $x = \pm k$, where k is a constant; this being like the problem of the change of temperature inside a red-hot iron slab which is suddenly put into water. (F.D. § 2. 2.)†

Ex. (ii). $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$ might be made definite by having ϕ given round some enclosing boundary in the (x, y) plane, as in electrostatic problems where ϕ is the potential. (F.D. § 3. 1.)

IV. *Generalize the problem as far as can easily be done by a change of variables.* Thus, in Ex. (i) above, put $\Phi = a\phi$, $X = bx$, $T = b^2\sigma t$, where a and b are parameters. Then $\sigma \partial^2\phi/\partial x^2 = \partial\phi/\partial t$ becomes $\partial^2\Phi/\partial X^2 = \partial\Phi/\partial T$; while for boundary conditions put $\Phi = 1$ when $T = 0$ and $\Phi = 0$ when $X = \pm\frac{1}{2}$. It is this simplified system in Φ, X, T which is solved by arithmetic. (See F.D. § 2. 2.) The arithmetical table of Φ as a function of X, T will then represent a threefold infinity of solutions as between ϕ, x, t, according to the values assigned to the parameters a, b, σ.

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* See (i) *The Calculus of Observations,* by Whittaker and Robinson, ch. xiv. (Blackie & Sons, Ltd.) (ii) *Practical Mathematical Analysis,* by Sanden. (Methuen, 1923.) (iii) Piaggio's *Book on Differential Equations.* (iv) Also various scattered papers by Runge, Huen, Kutta and N. Kryloff.

† References are given briefly thus to publications by L. F. Richardson:

F.D. refers to *London Roy. Soc. Phil. Trans.*, A 210, pp. 307-357.
S.S.U. refers to *London Roy. Soc. Phil. Trans.*, A 223, pp. 345-382.
W.P. refers to *Weather Prediction by Numerical Process* (Camb. Univ. Press).

Ex. (iii). Similarly, for the motion of a sphere in a medium resisting as the square of velocity, we solve by arithmetic this set of equations (S.S.U. p. 376):

$$dR_X/dT = -R_X\,|\sqrt{R_X^2+R_H^2}\,|\,,$$

$$dR_H/dT = -R_H\,|\sqrt{R_X^2+R_H^2}\,|-1\,;$$

with the further condition, say, that when $T=0$, $R_H=0$, $R_X=0{\cdot}5$, in which physical constants such as the diameter and mass of the sphere and the acceleration of gravity are not shown: they are brought in afterwards by a change of variables. (S.S.U. § 4. 2.)

V. *Replace differentials by finite differences*, taking care in so doing to arrange that the problem in differences tends to the given problem as its limit when the differences become infinitely small. Thus dy/dx is replaced by $(y_2-y_1)/(x_2-x_1)$, and the errors produced by this change are not considered until the difference-problem has been solved. (See Section XIV. below.) *It is very important that the differences should be "centered."* * To illustrate centering:

Ex. (iv). Suppose that we are given that $dy/dx=y$, and that y_1, y_2, y_3, y_4 are discrete values of y at intervals h of x, then dy/dx is replaced by $(y_2-y_1)/h$ and is considered as existing midway between the values of y from which it is derived. Thus, in place of $dy/dx=y$, we work with $(y_2-y_1)/h=\frac{1}{2}(y_2+y_1)$. The uncentered differences, which we must avoid, are those in which the second member of this equation would be replaced by either y_1 or y_2.

VI. *Find out whether the problem belongs to the "Marching" or to the "Jury" class.* A "*marching*" *problem* is one in which the integral can be stepped out from a part of the boundary. Thus in Ex. (iv), if we are given that $y_1=1$, say, then y_2, y_3, y_4, etc., can be found in succession, for y_2 must satisfy

$$(y_2-1)/h=\tfrac{1}{2}(y_2+1),$$

and so on.

Again, Ex. (iii) is a marching problem, for the whole courses of R_X and R_H as functions of T are fixed by their given values at $T=0$.

Ex. (i) also is a marching problem, for the value of Φ can be stepped out from $T=0$ by successive steps of δT.

The prediction of astronomical events belongs to the marching class. Also weather prediction belongs to the same (W.P. p. 3), at least if you believe that future weather is determined by present weather together with astronomical events which are foreknown.

A "*jury*" *problem* is one such that the integral at any particular set of values of the independent variables must be determined by reference to all parts of boundary considered together, just as the verdict has to satisfy all the jurymen seated round the table. For example,

Ex. (v), $d^2y/dx^2=3$ with $y=0$ when both $x=0$ and $x=1$. Here the integral cannot be stepped out from either $x=0$ or $x=1$. Again, in Ex. (ii), the potential ϕ at a point in the field is affected jointly by the potentials of all the conductors.

Ex. (vi). It is a jury problem to find the gravest mode of vibration of a plate with clamped edges. (F.D. pp. 316 to 318.)

Ex. (vii). Another jury problem is to find the stresses inside an elastic solid when the stress on the surface is given (F.D. § 4); for looking at the problem physically, it is evident that the stress at any point inside would, in general, be affected by the stress on any and every part of the surface.

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* See W. F. Sheppard, "Central Difference Formulae," *Proc. Lond. Math. Soc.* xxxi. pp. 449 to 488 (1899).

VII. *Solubility.* All *marching problems* with real variables are soluble by arithmetic, and the procedure is usually fairly simple. (F.D. § 2. 1.)

The routine toil increases with the order, the degree, and the number of variables, but these changes require hardly any new ideas, in contrast with the great complications which arise in analysis.*

Jury problems can always be converted into a set of algebraic simultaneous equations of the same degree, and so are soluble if linear—for instance, Ex. (vi). But the toil increases rapidly with the number of equations.

Alternatively, jury problems, in real variables, are soluble by successive approximations (F.D. § 3. 2) if they are linear in the dependent variable and its derivatives, and if, in addition, there exists a one-signed homogeneous quadratic function of the dependent variable and its derivatives, which is stationary when and only when the given differential equation is satisfied. This function is often potential energy. Thus, in Ex. (ii), the electrostatic energy is stationary for infinitesimal disturbances from the actual distribution. Again, in Ex. (vi), the elastic energy is stationary. The formal investigation of these matters is set out in F.D. Appendix, a development of a suggestion due to A. E. H. Love.

VIII. *Method of solution of marching problems, having a single independent variable.* Arrange the work in columns, giving the first column to the independent, the next to the dependent, and subsequent columns to its differential coefficients as far as these are needed.

Ex. (viii). Solve $\frac{d^2y}{dx^2}+e^{-x^2y^2}=0$, together with $y=1$ when $x=0$ and $dy/dx=0$ when $x=-0{\cdot}05$. We put instead $\delta^2y/\delta x^2=-e^{-x^2y^2}$. In the table below, the given value of y determines $\delta^2y/\delta x^2$ to be $-e^{-0^2 1^2}=-1$ in the position marked a in the adjoining table. Next a, when multiplied by δx and added to the given value of $\delta y/\delta x$, determines the number b. Next $b\,.\,\delta x$ added to the given value of y determines the number c, and one cycle of operations is complete. The next cycle begins when c is used to determine the number marked a_1, thus $a_1=-e^{-(0{\cdot}1c)^2}$. Thus the process can be continued quite easily, and without coming to a stop.

x	y	$\delta y/\delta x$	$\delta^2y/\delta x^2$
$-0{\cdot}1$			
$-0{\cdot}05$		0·000 given	
0·0	1·000 given		$-1{\cdot}000=a$
0·05		0·900$=b$	
0·1	1·090$=c$		$-1{\cdot}089=a_1$
0·15		0·791	
0·2	1·169		

IX. *Marching problems with more than one independent.* Ex. (i), which contains two independents, is worked out in F.D. § 2. 2. Many problems contain the four independents of space-time, and are correspondingly laborious. But much the same ideas suffice for their treatment. (See W.P.)

X. *The first few steps in a marching problem are often difficult to centre,* and yet if they are not centered, serious errors may arise. Various devices are available to put this in order—for instance, "the algebraic first step." To illustrate this device, suppose that in Ex. (viii) the value of dy/dx was given

* Vide *Differential Equations,* by Forsyth. (Macmillan.)

at $x=0{\cdot}00$ instead of at $x=-0{\cdot}05$. Then there would be no number suitably placed to which $a\,.\,\delta x$ could be added, and, at first sight, we could not begin. However, introduce the two algebraic unknowns f and g in the table below:

x	y	$\delta y/\delta x$	$\delta^2 y/\delta x^2$
$-0{\cdot}1$			
$-0{\cdot}05$		f	
$0{\cdot}0$	$1{\cdot}000$ given		$-1{\cdot}000=a$
$0{\cdot}15$		g	
$0{\cdot}1$			

Then $\frac{1}{2}(f+g)$ equals the given value of $\delta y/\delta x=0{\cdot}95$, say. Whereas

$$(g-f)/0{\cdot}1=a=-1{\cdot}000.$$

This pair of equations determines both f and g, and afterwards the arithmetic goes on merrily without any further need of assistance from algebra.

Other devices for making the first step are described in W.P. ch. 7/2. The most universally applicable device is to make a few initial steps uncentered, but much shorter than the rest.

XI. *The solution of jury problems by simultaneous equations.* Suppose that we have to solve $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$, with given values of ϕ on a closed boundary. Let us imagine that we have numbers arranged in rows and columns to represent values of ϕ in the (x, y) plane. For simplicity make δx, the interval between successive columns equal to δy, the interval between successive rows. A small portion of this numerical table is represented algebraically for the sake of argument by the letters p, q, r, s, t, arranged as shown. On the horizontal row we have first differences $(q-p)$ and $(p-s)$. The second difference is therefore $(q-p)-(p-s)=q-2p+s$, and is centered at the same point as p. Thus we replace $\partial^2\phi/\partial x^2$ by $(q-2p+s)/(\delta x)^2$. Similarly, we replace $\partial^2\phi/\partial y^2$ by $(r-2p+t)/(\delta y)^2$. And since $\delta x=\delta y$, it follows that $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2$ is replaced by $(q+r+s+t-4p)/(\delta x^2)$. For brevity let us call this $\Delta\phi/(\delta x)^2$ so that $\Delta\phi$ at any point of the table is computed by the following rule:

$$\begin{array}{ccc} & r & \\ s & p & q \\ & t & \end{array}$$

"Add together the numbers at the four nearest surrounding points, and subtract four times the number at the point taken." This difference, which we have called $\Delta\phi$, has by hypothesis to vanish.

Suppose that the given boundary, at which ϕ is known, is an irregular octagon, and that, to keep everything workable, we use very few points, so that the boundary appears thus in the (x, y) plane. That is to say there are eight outside points, called "boundary points," at which ϕ has the numerical values shown. Enclosed by these are four other points, called "body-points," at which we seek to determine ϕ so as to satisfy $\Delta\phi=0$. Give the unknowns the symbols a, b, c, d in the situations shown. Then by the vanishing of $\Delta\phi$ at the body-point a we have

$$\begin{array}{cccc} & 2 & 3 & \\ 1 & a & b & 4 \\ 0 & d & c & 0 \\ 0 & 0 & & \end{array}$$

$$1+2+b+d-4a=0,$$

and similarly at the other three body-points:

$$a+3+4+c-4b=0,$$
$$d+b+0+0-4c=0,$$
$$0+a+c+0-4d=0.$$

On solving these four simultaneous equations, we find

$$a=1{\cdot}4583,\quad b=2{\cdot}2917,\quad c=0{\cdot}7083,\quad d=0{\cdot}5417,$$

which gives the distribution of ϕ for these coarse differences. But if we desired more accuracy we should have to employ smaller differences and to find more unknowns, with a corresponding increase of toil.

As another illustration, the largest period of oscillation of a square plate, will be found worked out by this method in F.D. pp. 316 to 318.

XII. *The solution of jury problems by successive approximations.* As many as 114 unknown body-values have been determined by this method (F.D. p. 343). The toil was long, but most of it was a mere routine, in comparison with which the solution of 114 simultaneous equations would have seemed a dreadful task. For economy in printing the process will be illustrated here in miniature, by solving the problem of the preceding section, in which there are only four unknowns. Suppose then, that we have not solved the simultaneous equations and do not know a, b, c, d. Instead, we make a guess at them. We shall see that any four numbers do to begin with. In this respect the process resembles the children's game, "Think of a number, double it ...," in which you arrive at the same answer from whatever number you start. Try the guess set out near by, calling the distribution ϕ_1. Next work out $\Delta\phi_1$.

		2	3			
1	1	2	4		1·2	0·3
0	0·2	0·3	0		0·5	1·0
		0	0			
		ϕ_1			$\Delta\phi_1$	

$\Delta\phi_1$ exists at the body-points, but not at the boundary points.

Now form $\phi_2 = \phi_1 + \frac{1}{4}\Delta\phi_1$.

		2	3			
1	1·3	2·075	4		0·2	0·55
0	0·325	0·55	0		0·55	0·2
		0	0			
		ϕ_2			$\Delta\phi_2$	

Thus ϕ_2 has the same boundary values as ϕ_1, but different body-values. Now compute $\Delta\phi_2$. Then take a half of $\Delta\phi_2$ and add it to ϕ_2, forming $\phi_3 = \phi_2 + \frac{1}{2}\Delta\phi_2$. Then compute $\Delta\phi_3$.

		2	3			
1	1·4	2·35	4		0·35	−0·35
0	0·6	0·65	0		−0·35	0·35
		0	0			
		ϕ_3			$\Delta\phi_3$	

Take one-sixth of $\Delta\phi_3$ and add it to ϕ_3, thus forming $\phi_4 = \phi_3 + \frac{1}{6}\Delta\phi_3$.

It is then found that, to the accuracy permitted by the digits taken, $\Delta\phi_4$ vanishes in all four body-points, so that ϕ_4 is the required solution. On

looking back it is seen that ϕ_4 agrees with the solution of the simultaneous equations:

	2	3			
1	1·4583	2·2917	4	0	0
0	0·5417	0·7083	0	0	0
	0	0			
	ϕ_4			$\Delta\phi_4$	

The reader may like to see that this result is not a fluke, by taking in ϕ_1 his own choice of the four body numbers, and repeating the operations with them.

The success of the process depends on the fact that the errors of the initial guess can be expressed in terms of "normal functions," P, each of which satisfies $\Delta P = -kP$, where k is positive and independent of position.

They are P_1, P_2, P_3, P_4, specified thus:

	0	0			0	0	
0	1	1	0	0	1	−1	0
0	1	1	0	0	1	−1	0
	0	0			0	0	
	$\Delta P_1 = -2P_1$				$\Delta P_2 = -4P_2$		

	0	0			0	0	
0	−1	−1	0	0	1	−1	0
0	1	1	0	0	−1	1	0
	0	0			0	0	
	$\Delta P_3 = -4P_3$				$\Delta P_4 = -6P_4$.		

Any arbitrary set of errors could, if known, be expressed as the sum of multiples of P_1, P_2, P_3, P_4. For instance:

	0	0			
0	1	2	0	equals	$\frac{5}{2}P_1 + P_3 - \frac{1}{2}P_4$.
0	4	3	0		
	0	0			

The possibility of such expressions in general is discussed in F.D. Appendix.

So if ϕ_u is the distribution which we seek, namely, such as to satisfy $\Delta\phi_u = 0$, then our erroneous guess ϕ_1 is

$$\phi_1 = \phi_u + A_1P_1 + A_2P_2 + A_3P_3 + A_4P_4, \qquad (1)$$

where the coefficients of P_1, P_2, P_3, P_4 are unknown numbers, which may be called the amplitudes of the normal modes.

It follows that

$$\Delta\phi_1 = -2A_1P_1 - 4A_2P_2 - 4A_3P_3 - 6A_4P_4. \qquad (2)$$

Next we formed

$$\phi_2 = \phi_1 + \tfrac{1}{2}\Delta\phi_1. \qquad (3)$$

But it follows from (1) and (2) that

$$\phi_2=\phi_a+\tfrac{1}{3}A_1P_1-\tfrac{1}{3}A_4P_4. \qquad (4)$$

So that two of the amplitudes have been reduced to zero.

In general, our operation is

$$\phi_{n+1}=\phi_n+(\Delta\phi_n)/a, \qquad (5)$$

and this reduces to zero in ϕ_{n+1} the amplitude of any normal mode P, for which $a=k$, where k is the positive constant in $\Delta P=-kP$.

Thus, in the above example, by taking a in turn to be 2, 4, 6 in any sequence, we reduce the amplitudes of all four normal modes to zero, and so leave a result free from error. The single operation with $a=4$ removes two normal modes.

In general the process is not so slick, because we do not know the values of k, and there are many of them. It is, however, possible to form a rough estimate of the greatest and least of the numbers k (F.D. pp. 320, 321), and this estimate suffices. For if $a>\frac{1}{2}k_L$, where k_L is the greatest of the numbers k, then the operation (5) will reduce the absolute value of the amplitude of every normal mode. So that, by repeating the operation a sufficient number of times with this same value of a, it is possible to make the errors as small as we please. This was tried on the above example. There $k_L=6$. The number used was $a=4$, which exceeds $\frac{1}{2}k_L$. It was found that the errors of ϕ were less than 0·01 when ϕ_7 was attained.

It saves time, however, to spread out the values of a over the range covered by k. (See F.D. pp. 320 to 322.) Except when the values of k are accurately known, he arrives quickest whose guess at ϕ_1 is best.

XIII. *Errors due to finite differences.* For both marching and jury problems it is shown (F.D. § 1. 2) that the error is of the form $Ah^3+Bh^4+Ch^5+$etc., where h is the difference of the independent variables, supposed the same for all, and where A, B, C are functions of these variables. The odd powers of h are absent because the differences are centered. We repeat then the solution of the problem for different values of h. If h is sufficiently small only the first term of the series will matter.

Thus, if we make two solutions, the coarse and the fine, the latter having h half as big as in the former, then the error of the fine solution will be $\frac{1}{4}$ that of the coarse, and therefore the limit $h\to0$ will be obtained by moving $\frac{1}{3}$ as far on one side of the fine solution as the coarse solution is on the other side.

More generally, the term Bh^4 may matter; but this can be investigated by making another solution, very coarse or very fine. (See F.D. pp. 311, 317, 318.)

XIV. *Discontinuities.* At discontinuities this series $Ah^3+Bh^4+\ldots$may not be convergent. There are, so to speak, in the mathematical country, precipices and pit-shafts down which it would be possible to fall; but that need not deter us from walking about. Yet if we wish to explore these steep descents, pedestrianism must be supplemented by the acrobatics of the pure mathematician.

L. F. RICHARDSON.

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319. Of itself an arithmetic average is more likely to conceal than to disclose important facts; it is of the nature of an abbreviation, and is often an excuse for laziness.—A. L. Bowley, *The Nature and Purpose of the Measurement of Social Phenomena.*

320. He was endowed by nature with plenty of ability; though he had no training whatever in that branch of learning, he possessed something like a genius for mathematics, solving difficult problems in algebra, geometry, and trigonometry, provided that "the rule or theorem had been carefully read aloud to him."—*Athenaeum*, 1900, on *Life of General N. B. Forrest*, by J. A. Wyeth.

PRACTICAL NON-EUCLIDEAN GEOMETRY.

By Capt. T. C. J. Elliott, R.A.O.C.

1. In order to understand the meaning to be attached to Non-Euclidean Geometry and the use of the model measuring tape described below, it is advisable to say a few words upon Euclidean Geometry.

2. By a "*space*" we mean a discrete manifold. In a space of two dimensions the points can be classified by one of the two respects in which they differ, and cross classified by the other. Similarly, in a space of three dimensions, the points, or classified things, differ in three respects. *Distances* along a dimension are compared by counting point intervals, which are "equal" by definition. It is also by definition that counts in one dimension are "equal" to a count of the same number in another dimension, and that any two dimensions are "at right angles." By a "*line*" is meant an order of "points," and if each point has the same position relative to the preceding one, the line is defined as "*straight*." This relative position is also the "direction" of the lines. Two straight lines which have the same direction are defined to be "*parallel*."

3. There is much to be said for making the discrete manifold rather than physical experiments with optical lines, stretched strings, metre rules, and wooden models the foundation of school mathematics, a view put forward by the writer in *Classification and Mathematics*, Oundle, 1909, and in later writings. The geometry of the manifold is Euclidean, and such disadvantages as the impossibility of describing a circle are compensated for by the presence of natural axes of coordinates, namely, the dimensional directions. When "point," "straight line," "equal" and "right angle" have the usual physical meanings, the theory of a space of two dimensions applies (both with old and new definitions) to the points of a sheet of squared paper. The intersections of the two systems of equal-spaced lines at right angles form a discrete duplex.

4. The geometry of a discrete manifold being independent of the physical positions of its points, it retains its Euclidean character always. For example, if we make a classificatory model of a two-dimensional manifold, its geometry is the same whether we arrange the "points" in columns and rows, or mix them up anyhow.

It may therefore appear that the geometry of manifolds is no more promising than the old physical geometry as a path to Non-Euclidean Geometry. From the time of Euclid, about two thousand years elapsed before the faint clue afforded by the fifth postulate was successfully followed up.

5. In the case of the discrete manifold we have at any rate this additional clue, that the distance between two points is entirely a matter of counting, their distance in feet playing no part whatever. Therefore we are free to make suppositions about the *relative* correspondence of distances in two spaces without being hampered by the thought that we are disturbing the Euclidean character of either. In fact, the nature of the relation between distances which we propose to study is such that if a space A is taken as Euclidean, then a space B may be (relatively) Lobatchewskian; but if B is taken as Euclidean, then A is (relatively) Riemannian.

Both spaces are Euclidean, but distances have the relation that from *any* point in B consecutive steps *increase relatively* to A; or (what means the same) from *any* point in A consecutive steps *decrease* relatively to those in B.

6. By analogy with physical geometry we may express the relationship by saying that, to obtain distances in B, judged from the standpoint of A, we must use a ruler graduated thus:

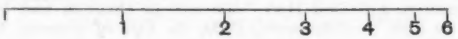

Fig. 1.

If the ruler was long enough the steps would become infinitely small. But, to obtain distances in A, judged from the standpoint of B, we must use a ruler graduated thus :

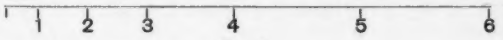

FIG. 2.

which may be regarded as the former ruler numbered from the other end.

For convenience in applying to the circumference of a circle, a tape about a yard long graduated from both ends is better than a straight ruler.

In practical work we naturally take ordinary physical space as the Euclidean one, and consider the results of substituting for a metre ruler the two kinds of ruler described above.

7. *Practical Work.*

(1) The first thing to observe is that if A, B, and C are three points in order, then the distance from A to C is no longer the sum of AB and BC.

(2) The smaller the distance which we take, the closer do the measurements agree with the ordinary metre ruler.

(3) If we compare the ratio of circumference to diameter of various circles we find that for Fig. 1 the ratio is greater than π, and increased with size of circle.

For Fig. 2 the reverse holds.

(4) If we measure the perimeter of a square $ABCD$ as compared with a diameter EF, we see that the distance AB, which is a quarter of the perimeter,

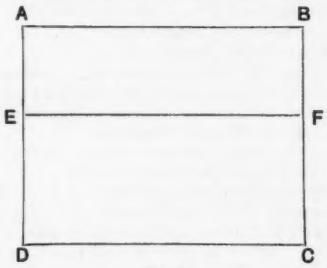

FIG. 3.

must be greater than EF for Fig. 1 but less than EF for Fig. 2. Thus we are led to the two cases of Saccheri.

(5) The hypotenuse of a right-angled triangle is no longer the arithmetic sum of the squares of the arms, but is greater than the sum for Fig. 1 and less for Fig. 2.

(6) In the matter of parallels, as in other cases, experiment suggests, but does not prove, results.

In the case of Fig. 1 :

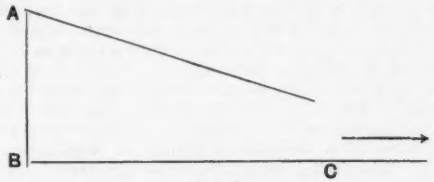

FIG. 4.

If the point C moves to infinity in the direction of the arrow, its steps continually increase, and finally the angle at C tends to zero, while the angle at A becomes a finite angle less than a right angle.

In the case of Fig. 2 the steps of the point as it moves to infinity become shorter and shorter, and it is unable to go beyond a finite distance from B. The angle at C is finite, while the angle at A approaches a right angle. Relative to the Euclidean space, the second space is like the surface of a sphere, the ultimate limit BC being the quadrantal arc. All "straight" lines from the final position of C on to AB are at right angles to AB.

(8) Certain theoretical questions, easier to raise than to solve, are suggested by the use of the ruler instead of the historical manner of approaching Non-Euclidean Geometry.

(1) Suppose we have *two* rulers of the type of Fig. 1, but differently graduated, and we use one for one dimension of a space, and the other for another dimension. Of what nature is the geometry ?

(2) Suppose we use a ruler of type Fig. 1 of one dimension, but type Fig. 2 for another. Or suppose we simply allow the other to remain Euclidean. Of what nature is the geometry ?

(3) If a space B is, for example, Riemannian to C, and A is Riemannian to B, what is the geometry of A relative to C ? T. C. J. ELLIOTT.

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321. Suggested epitaphs on Albert Einstein.

"Here Einstein rests, if any rest there be
For souls obsessed by relativity.
In life his axe shone ever at the root
Of some proud thought-encumbering Absolute.
Death holds him now, but ere his force was spent,
Space shrank her bounds, Light, at his bidding, bent,
And Gravitation fled the firmament.
Nor spared he Man: one arrow from his bow
Pierced Newton, one transfixed Galileo;
Their gold he proved with dubious ore alloyed,
And threatened Ether shuddered in her void;
With shattered axioms Nature's realm was strown;
Till Death, incensed at powers that paled his own,
Shook the Imperial Thinker from his throne."

—THOMAS THORNELY.

"Beyond all discord, all dispute,
Lies he who smote the Absolute;
O human race, long as you live,
Be proud of this your Relative."—CONAR.

—*The Spectator*, April 4, 1925.

322. "When it is considered that the present Subject is of a Nature the most exalted that the human Mind ever conceived—that it is the most noble invention, antient or modern, that ever was obtained by the researches of Man—that its Inventor, Sir ISAAC NEWTON, by it has immortalized our Country, and, at the same time, has given a Clew to obtain Truths of so sublime a nature as no other method of reasoning could lead to;—that this Work is the most elementary and compleat of any ever written—that it has been long out of print, and almost unobtainable from its Scarcity and price—the Publishers conceive, that, for what they have here done, an Apology is unnecessary.—Advertisement, prefixed to the second edition of MacLaurin's *Treatise on Fluxions*, 1801." Composed, presumably, by William Davis. [Per Prof. E. H. Neville.]

AN APPLICATION OF BESSEL FUNCTIONS TO A PROBLEM IN OPTICAL RESOLUTION.*

By Arnold Buxton, M.A.

I. *An Outline of the Problems.*

Whatever theory is finally adopted for the propagation of light, there will always remain the interesting fundamental problem as to how far an object is exactly reproduced by an optical system. In this paper, the enquiry will be considered in its simplest form, and only the simplest types of objects can be considered. Taking the wave theory, let us consider the reproduction of *an infinitely distant self-luminous point source*, such as a star, through an optical system, such as a telescope. Sir George Airy investigated this case in 1834; he shewed what W. Herschel had recorded in 1782, that the image in the focal plane † of the optical system consisted of a central concentration of light, surrounded by light rings of rapidly diminishing intensity (see Fig 2), and gave formulae to shew the falling off in intensity of the light away from the centre of the image. This disc of light, then, is the approximation to the point-image of geometrical optics. The deformation is due to the distortion of the sphericity of the light waves making the image by diffraction at the aperture of the system. As in the propagation of wireless waves, however, diffraction is not the only cause of distortion.

The waves are also affected by the inherent defects of spherical lenses known as aberrations. Klein has shewn that there can be no perfect optical system, and the residuals of these defects must affect the reproduction of the point-source. Airy neglected this aspect, as also did Lommel, who extended Airy's work to the finitely distant point-source, and at the same time expressed the results in terms of the Bessel Functions. Lord Rayleigh, however, discusses the effects of two defects in his article on *Diffraction* in the *Encyclopaedia Britannica*. A complete treatment of the problem should aim at the determination of the distribution of the light in the various reproductions of the point-source. The method is essentially one of comparison, and taking the Airy reproduction as the standard (diffraction only being considered), we can determine the relative merits of other reproductions due to the presence, separately or in combination, of the aberrations. Then, agreeing on a basic definition as to how near two point-sources may be together and be seen, comparisons can be made of the resolution in each reproduction. With the extension of this work to the cases (1) when the incident parallel light from the infinitely distant self-luminous point-source is inclined to the axis of the optical system; (2) when the point-source is at a finite distance from the system, we have a complete treatment of the problem for the point-source. Regarding the self-luminous line-source as an aggregate of such point-sources, and a plane-source as an aggregate of such line-sources, the investigations can be usefully extended still further.

II. *How the Bessel Functions arise in Airy's Problem.*

Let Ox represent the axis about which the optical system is symmetrical. Let $PP'Q$ represent the aperture (Fig. 1) with its plane perpendicular to Ox, and Oy, Oz as perpendicular radii.

Let $y'Fz'$ be the focal plane and Fig. 2 represent the configuration of the light distribution in this plane. We shall first consider the intensity of the light at F_1, distant y_1 from the centre of the image—and F_1 from symmetry can be taken on Fz'—as compared with that at F. Now the effect of

* A paper read at the Annual Meeting of the Mathematical Association, January 6, 1925.

† See C. Pendlebury's *System of Lenses*.

diffraction on the spherical wave is to deflect the normal ray from its geometrical course PF to PF_1, so that the wave has to travel an extra distance

$$PF_1 - PF.$$

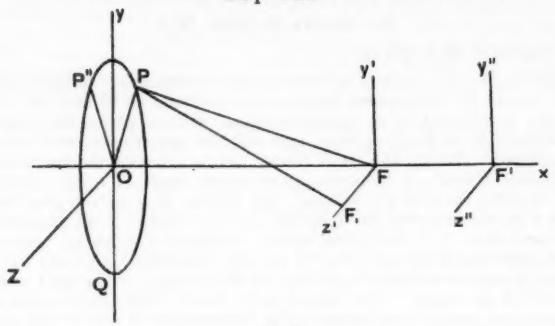

FIG. 1.

If $OP = r$ and $\angle POy = \phi$, then the coordinates of P are $0,\ r\cos\phi,\ -r\sin\phi$, and $(f, 0\ y_1)$, $(f, 0, 0)$ are the points F_1 and F respectively.

Then $$PF_1 - PF = \sqrt{r^2\sin^2\phi + r^2\cos^2\phi + 2ry_1\sin\phi + y_1^2 + f^2} - \sqrt{r^2+f^2}$$
$$= \frac{ry_1\sin\phi}{f}.$$

FIG. 2.

Or more fully, from Fig. 3,

$$PF_1^2 = PP_0^2 + P_0F_1^2$$
$$= PP_0^2 + P_0F_{1\cdot0}^2 + F^1F_{1\cdot0}^2$$
$$= PP_0^2 + P_0F_{1\cdot0}^2 + OF^2$$
$$= r^2\cos^2\phi + (r\sin\phi + y_1)^2 + f^2$$
$$= r^2 + 2ry_1\sin\phi + f^2$$
$$= (r^2+f^2)\left(1 + \frac{2ry_1\sin\phi}{r^2+f^2}\right),$$

i.e.,
$$PF_1 = \sqrt{r^2+f^2}\left(1 + \frac{2ry_1\sin\phi}{f^2+r^2}\right)^{\frac{1}{2}}$$
$$= \sqrt{r^2+f^2}\left(1 + \frac{ry_1\sin\phi}{f^2+r^2}\right), \text{ neglecting squares and higher powers of } y_1,$$
$$= PF + \frac{ry_1\sin\phi}{\sqrt{f^2+r^2}}$$
$$= PF + \frac{ry_1\sin\phi}{f},$$

neglecting $\frac{r^2}{f^2}$ and higher even powers of $\frac{r}{f}$ which multiply $\frac{ry_1\sin\phi}{f}$;

i.e.,
$$PF_1 - PF = \frac{ry_1\sin\phi}{f}.$$

Hence, if the vibration of the secondary wave at P can be denoted by $y=a \sin wt$ when the ray is directed from P to F, it will be denoted by

$$y=a \sin\left(wt+\frac{ry_1 \sin \phi}{f}\right)$$

for the bent ray PF_1, and

$$y=a \sin\left(wt-\frac{ry_1 \sin \phi}{f}\right)$$

for the corresponding secondary wave from P'', where OP'' makes angle $-\phi$ with Oy. Compounding these two vibrations, the resultant vibration will

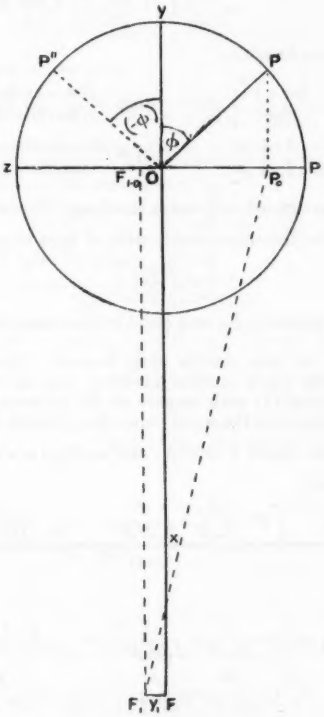

FIG. 3.

have as amplitude $a \cos \frac{(ry_1 \sin \phi)}{f}$. Integrating over a thin circular strip of radius r, the resultant amplitude at F_1 is proportional to

$$\int_0^{\frac{\pi}{2}} a \cos \frac{(ry_1 \sin \phi)}{f} d\phi.$$

The intensity at F_1 is equal to the square of the amplitude, and hence the intensity at F_1 due to a thin circular strip is proportional to

$$\left[\int_0^{\frac{\pi}{2}} \cos\left(\frac{ry_1}{f}\sin\phi\right)d\phi\right]^2.$$

Writing $\frac{ry_1}{f}=z$,

$$I=\int_0^{\frac{\pi}{2}} \cos\left(\frac{ry_1}{f}\sin\phi\right)d\phi=\int_0^{\frac{\pi}{2}} \cos(z\sin\phi)\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\left[1-\frac{z^2\sin^2\phi}{\underline{2}}+\frac{z^4\sin^4\phi}{\underline{\phi}}-\ldots\right]d\phi.$$

Using the reduction formula,

$$\int_0^{\frac{\pi}{2}}\sin^{2m}\phi\,d\phi=\frac{2m-1}{2m}\int_0^{\frac{\pi}{2}}\sin^{2m-2}\phi\,d\phi=\frac{(2m-1)(2m-3)\ldots3\,.\,1\pi}{2m(2m-2)\ldots4\,.\,2\,.\,2},$$

$$I=\frac{\pi}{2}\left\{1-\frac{z^2}{2\,.\,\underline{2}}+\frac{3\,.\,1}{2\,.\,4}\frac{z^4}{\underline{4}}-\ldots+(-1)^m\frac{(2m-1)(2m-3)\ldots1}{2m(2m-2)\ldots2}\frac{z^{2m}}{\underline{2m}}\ldots\right\}.$$

This latter series is defined as $\frac{\pi}{2}$ times the Bessel Function $J_0(z)$, so that the intensity at F_1 due to the thin circular strip of light is proportional to

$$\left[J_0\left(\frac{ry_1}{f}\right)\right]^2. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By varying y_1 the intensity for any point in the image can be obtained from the Bessel tables.

Now the area of the thin circular strip is $\pi\,dr^2$. Hence the proportional intensities due to the whole circular aperture (say of radius unity) can be obtained by integrating (1) with respect to dr^2 between the limits 0 and 1. In other words, we integrate the series value of $J_0(z)$ term by term with respect to $\frac{dz^2}{(y_1/f)^2}$ between the limits 0 and $\frac{y_1^2}{f^2}$, and as $J_0(z)$ is a series of even powers of z, we obtain that

$$\text{intensity at } F_1=\frac{\left\{\int_0^{y_1^2/f^2}\left[z^2-\frac{z^4}{2\,.\,2^2}+\frac{z^6}{3.2^2.4^2}\cdots+\frac{(-1)^m z^{2m+2}}{(m+1)2^2\,.\,4^2\ldots(2m)^2}\cdots\right]\right\}^2}{y_1^4/f^4}.$$

Put $\mu=\frac{y_1}{f}$; then

$$\text{intensity at } F_1=\left[\mu^2-\frac{\mu^4}{2\,.\,2^2}+\frac{\mu^6}{3.2^2.4^2}\cdots+\frac{(-1)^m\mu^{2m+2}}{(m+1)2^2.4^2\ldots(2m)^2}\cdots\right]^2\Big/\mu^4$$

$$=\left[1-\frac{\mu^2}{2\,.\,2^2}+\frac{\mu^4}{3.2^2.4^2}\cdots+\frac{(-1)^m\mu^{2m}}{(m+1)2^2\,.\,4^2\ldots(2m)^2}\cdots\right]^2.$$

The series

$$\frac{1}{2}\left\{\mu-\frac{\mu^3}{2\,.\,2^2}+\frac{\mu^5}{3.2^2.4^2}\cdots+\frac{(-1)^m\mu^{2m+1}}{(m+1)2^2\,.\,4^2\ldots(2m)^2}\cdots\right\}$$

is defined as $J_1(\mu)$, where J_1 is the Bessel Function of the first order.

Hence the intensity at $F_1=\left[\frac{2J_1(\mu)}{\mu}\right]^2=\left\{\frac{2J_1\left(\frac{y_1}{f}\right)}{y_1/f}\right\}^2$. $\qquad\ldots\ldots\ldots\ldots\ldots(2)$

From (1) the intensity at the centre (*i.e.* where $y_1=0$) is unity, and so also for (2). Hence the two reproductions can be compared by suitable variations of y_1. The zeros and maxima of (1) and (2) give the radii of the dark and light rings respectively. These are all given in the Bessel tables.

Incidentally, the maxima of (1) are the roots of (2), and the maxima of (2) are the zeros of $J_2(\mu)$.

The light received within the circle of radius y_1 is proportional to

$$4\pi f^2 \int_0^{y_1} \frac{J_1^2(\mu)}{\mu^2}\, d\mu^2,$$

i.e. it is proportional to $\int_0^{y_1} \frac{J_1^2(\mu)}{\mu^2}\, d(\mu^2)$,

or to $\int_0^{y_1} \frac{J_1^2(\mu)}{\mu}\, d\mu = \int_0^{y_1} \frac{J_1(\mu)}{\mu} \cdot J_1(\mu)\, d\mu.$

$J_n(\mu)$ can be defined as the series

$$\frac{\mu^n}{2^n \lfloor n} \left\{1 - \frac{\mu^2}{2(2n+2)} + \frac{\mu^4}{2 \cdot 4(2n+2)(2n+4)} - \dots\right\},$$

and it is easy to shew that the differential of series value of (2) is a factor of $J_2(\mu)$ from this definition.

From the series for $J_0(\mu)$ the general term is

$$(-1)^m \cdot \frac{(2m-1)(2m-3)\dots 1}{2m \cdot (2m-2)\dots 2} \cdot \frac{\mu^{2m}}{\lfloor 2m}.$$

From the series for $J_1(\mu)$ the general term is

$$\frac{\frac{1}{2}(-1)^m \mu^{2m+1}}{(m+1)2^2 \cdot 4^2 \dots (2m)^2}.$$

General term in $J_0(\mu) - J_1'(\mu) = \frac{(-1)^m \mu^{2m}}{2^2 \cdot 4^2 \dots (2m)^2} - \frac{2m+1}{2}(-1)^m \frac{\mu^{2m}}{(m+1)2^2 \cdot 4^2 \dots (2m)^2}$

$$= \frac{(-1)^m \mu^{2m}}{2^2 \cdot 4^2 \dots (2m)^2} \left\{\frac{2m+2-2m-1}{2m+2}\right\}$$

$$= \frac{(-1)^m \mu^{2m}}{(2m+2)(2m)^2 \dots 4^2 \cdot 2^2} = \frac{\frac{1}{2}(-1)^m \mu^{2m}}{(m+1) \cdot 2^2 \cdot 4^2 \dots (2m)^2}$$

$$= \text{general term in } \frac{J_1(\mu)}{\mu}.$$

We can also test that $J_1(\mu) = -\frac{dJ_0(\mu)}{d\mu}$,

so that $\frac{J_1^2(\mu)}{\mu} = -J_0(\mu)J_1'(\mu) - J_1'(\mu)J_1(\mu).$

Hence $\int_0^{y_1} \frac{J_1^2(\mu)}{\mu}\, d\mu = \int_0^{y_1} \left[-J_0(\mu)J_0'(\mu) - J_1'(\mu)J_1(\mu)\right] d\mu$

$$= \left[-\tfrac{1}{2}\{J_0^2(\mu) + J_1^2(\mu)\}\right]_0^{y_1}$$

$$= \frac{1}{2}\left[1 - J_0^2\left(\frac{y_1}{f}\right) - J_1^2\left(\frac{y_1}{f}\right)\right].$$

Hence the fraction of the total light outside a circle radius y_1 is

$$J_0^2\left(\frac{y_1}{f}\right) + J_1^2\left(\frac{y_1}{f}\right),$$

and at a dark ring $J_1\left(\frac{y_1}{f}\right) = 0$, so that outside any dark ring $J_0^2\left(\frac{y_1}{f}\right)$ represents the fraction of the whole light, and, as Lord Rayleigh shewed, $\frac{9}{10}$ of the whole light is received within the second dark ring and 84 per cent. within the first.

Now, assuming that the eye can separate two point-sources very close together when the centre of one is on the circumference of the first dark ring of the other, and *vice-versa*, it follows that, since 3·83 is the first zero of $J_1\left(\frac{y_1}{f}\right)$, two infinite point-sources can be resolved when their angular separation is $\frac{3{\cdot}83\lambda}{2\pi r}$, where λ is the wave-length.

III. *The Wave-distorting Effects.*

In paragraph II. the effect of the aperture in bending the rays of the emergent light was considered. We have now to take into account the various deformations of the waves which occur irrespective of the aperture, although they vary with the size of the aperture.

(*a*) *Malfocus.* In the first place, let us investigate the image received not in the plane $y'Fz'$, but in a parallel plane $y''F''z''$ at a distance df from the former. The reproduction is now said to be affected with malfocus. From Fig. 4, the central part of the wave front travels a distance equal to AF', and so is ahead of the part from P by an amount equal to $AF' - PF'$

$$\begin{aligned}
&= PF + df - PF' \\
&= df(1 - \cos PF'A) \\
&= 2df \sin^2 \frac{PF'A}{2} \\
&= 2df \frac{r^2}{4f^2}, \text{ approximately,} \\
&= cr^2.
\end{aligned}$$

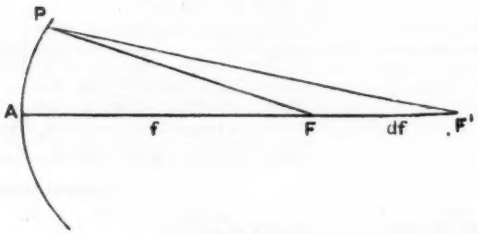

FIG. 4.

Hence, if we include diffraction as well, the vibration from P has now for its equation $y = a \sin\left(\omega t + \frac{ry_1}{f} \sin\phi - cr^2\right)$.

Taking this with the corresponding vibration from P'' (see Fig. 1), we have a resultant vibration which is compounded of two vibrations at right angles, whose amplitudes are proportional to

$$\cos\left(\frac{ry_1}{f} \sin\phi\right) \cos cr^2 \quad \text{and} \quad \cos\left(\frac{ry_1}{f} \sin\phi\right) \sin cr^2.$$

Integrating these for the whole aperture of radius unity, we have that the intensity is proportional to

$$\left[\int_0^1 \int_0^{\frac{\pi}{2}} \cos\left(\frac{ry_1}{f} \sin\phi\right) \cos cr^2 . \pi\, dr^2 d\phi\right]^2$$

$$+ \left[\int_0^1 \int_0^{\frac{\pi}{2}} \cos\left(\frac{ry_1}{f} \sin\phi\right) \sin cr^2 . \pi\, dr^2 d\phi\right]^2,$$

i.e. I is proportional to

$$\left[\int_0^1 J_0\left(\frac{ry_1}{f}\right)\cos cr^2 . \pi\, dr^2\right]^2+\left[\int_0^1 J_0\left(\frac{ry_1}{f}\right)\sin cr^2 . \pi\, dr^2\right]^2.$$

As will be true of the succeeding integrals, this formula cannot be manipulated with the same ease as the corresponding result in paragraph II., even with the aid of Bessel tables. Lommel, however, gave much attention to the above integrals, although in another connection. There is an account of his treatment of them in Gray and Mathews' *Treatise on the Bessel Functions*. His graphical method is especially interesting, and his curves can be readily adapted towards the determination of the zeros and maxima of I, thus solving the positions of the dark and light rings in the image. To find the intensity at any point in the reproduction, however, it will be necessary either to calculate the integrals by approximate integration (I have calculated about 300 of them by using $y=a+bx+cx^2+dx^3+ex^4$ as the approximation curve*), or to calculate a series of Bessel Functions given in Lommel's account. Let the first integral be C and the second S, then

$$dC=J_0\left(\frac{ry_1}{f}\right)\cos cr^2 . \pi\, dr^2,$$

$$dS=J_0\left(\frac{ry_1}{f}\right)\sin cr^2 . \pi\, dr^2,$$

i.e.
$$(dC)^2+(dS)^2=\left\{J_0\left(\frac{ry_1}{f}\right)\pi\, dr^2\right\}^2$$

and
$$\frac{dS}{dC}=\tan cr^2,$$

so that in the curve of C against S, the distance of the points on it from the origin represents the corresponding intensity, an element ds of the length of the curve is equal to $J_0\left(\frac{ry_1}{f}\right)\pi\, dr^2$, whilst the angle of slope of the tangent with the axis of C is equal to cr^2.

Since
$$ds=J_0\left(\frac{ry_1}{f}\right)\pi\, dr^2,$$

$$s=\frac{2J_1(z)}{z}, \text{ where } z=\frac{ry_1}{f}, \text{ from II.}$$

Hence, some of the zeros of C^2+S^2 will occur at the zeros of $J_1(z)$.

Lommel's analysis is, however, more complete, and he shews that the zeros and maxima are given by the zeros of a certain series of Bessel Functions and those of $J_1(z)$. To adapt his curves for our purpose would mean drawing curves of c against y_1, and labelling them $U=0$, together with lines parallel to the c-axis representing the values of y_1 which make $J_1\left(\frac{ry_1}{f}\right)=0$. The alternate intersections of these curves and lines by a line parallel to the y_1-axis corresponding to any given value of c would give the values of y_1, or the radii of the dark and light rings respectively. A useful feature of the graphs would be in the probable location of a malfocus image, where the central intensity was reduced and the remaining light distribution such that two point-sources could be seen even within the accepted limits.

(*b*) *Spherical Aberration.* Spherical aberration arises owing to the fact that the outer rays are focussed at a different point from the more central

* The corresponding rule appears in Dr. W. F. Sheppard's article on "Mensuration," *Encyclopaedia Britannica*, vol. 18.

rays which travel close to the axis. Hence the waves are again distorted by the presence of this defect. In Fig. 5, let the outer rays be focussed at F_2 and

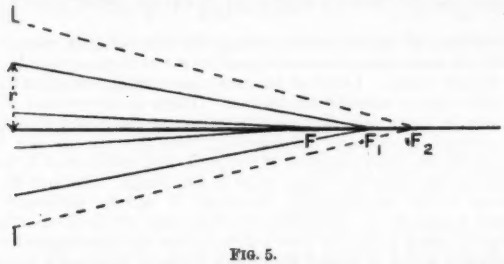

FIG. 5.

the axial rays at F; then an intermediate ray from a zone of radius r will arrive at F_1 say. The difference in the intermediate and axial paths will be proportional as in (a) to r^2 FF_1, and FF_1 itself can be shewn to be proportional to r^2, so that the difference in the paths of the rays in the presence of spherical aberration is proportional to r^4. Hence, if we write c_1r^4 instead of cr^2 in the formulae for malfocus, we shall have a measure of the effect of spherical aberration on the image in the plane $y'Fz'$ (see Fig. 1).

The integrals involved do not seem to have been treated in the same systematic manner as those in paragraph III. (a). The roots of $J_1(z)=0$ will have the same significance as before, and the required zeros and maxima can be obtained by interpolation from the calculated values of the intensities by approximate integration. The central intensities (*i.e.* when $y_1=0$) are examples of the well-known Fresnel integrals

$$\int J_{\frac{1}{2}}(z)dz \quad \text{and} \quad \int J_{-\frac{1}{2}}(z)dz.$$

Further, we can combine defects (a) and (b) by writing $cr^2+c_1r^4$ instead of cr^2 in the formulae in (a), and it can be shewn that the best image occurs when $c=-c_1$, *i.e.* when the plane is taken midway between F and F_2.

(c) The colour defects (chromatic aberration) can also be taken into account. The rays from P are split up according to the colour (*i.e.* the wave-length), and the foci are distributed along the axis as in Fig. 6. The problem, here, is the

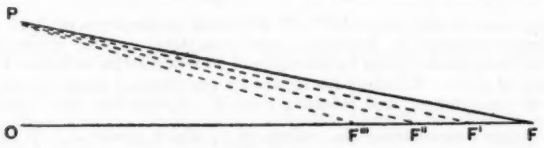

FIG. 6.

same as for malfocus, and the intensity formula is the same, except that c is now a function of λ, the wave-length, and so there will be double integration with respect to λ and r^2.

In conclusion, it may be stated that other defects occur when the objects are obliquely situated, so that the emerging rays are inclined to the axis of the system; also there are luminous lines and planes to be considered. Enough has been said, however, to shew the importance of the use of Bessel Functions not only in the concise expression of the results, but also to consolidate (with extra tables, if possible) the work for the complete solution of the problem.

A. BUXTON.

Obituary.

The Rev. H. C. WATSON.

The Editor has asked me to write something about the Rev. Henry Charles Watson, who was senior mathematical master at Clifton from 1870-1900, including the years 1874-79, which I spent there. The main facts about his peaceful life are soon told. He was a scholar of Trinity College, Cambridge, and graduated as 8th wrangler in 1865 when Strutt, afterwards Lord Rayleigh, was senior wrangler. After a time, at Elizabeth College, Guernsey, he came to Clifton in May, 1870, to fill the gap caused by the death of C. H. Cay, and remained in charge of the mathematics, as above stated, till December, 1900. In 1901 he was appointed by his old college Vicar of Gainford on Tees (near Darlington), and must have made almost an ideal village clergyman until his retirement, when he bought a house at Ripon. But there he died in his 83rd year on April 16 last. A long and well-spent life. It seems almost intrusive to say more; and yet in gratitude for much instruction which has helped me through life and for much kindness received outside our official relations, I must try to honour his memory by some attempt to recall his aims and methods, for the possible guidance of others. Moreover, H. C. Watson has a special claim to our remembrance in that he was (from 1878 to 1890) Treasurer of the Mathematical Association in its earlier incarnation. But one makes such attempts with more and more diffidence—diffidence due, I hope, less to inertia than to the realization, as experience gathers, of the difficulties besetting the life of a schoolmaster, which, when as boys we knew him, we so lightly regarded. Since I learned from Watson I have been a schoolmaster (for a few weeks) myself, I have in later years rushed in as inspector where the Board of Education was not yet allowed to tread; and I would fain make use of some of the sympathy thus acquired in interpreting Watson's life and work: but alas! the boyish memories insist on dominating the picture. It seems better to let them have their way and to write frankly as a boy back at Clifton,—a boy of whom Watson hoped great things and to whom he therefore devoted hours of his precious leisure. So we are together again, back in Watson's study to rectify some of the deficiencies of the text-books, which were not so many or so carefully considered in those days. Todhunter was one of the Moderators for Watson's Tripos, and Todhunter had written text-books * which were regarded as containing the whole duty of young mathematicians, or very nearly. But his devotees admitted that at least alternative proofs could be given in some cases, and these Watson provided with skill and precision. But first a pen must be selected and the accumulations of ink which had dried on it carefully scraped off with a pen-knife. Then when a few sheets of white paper had been neatly squared and adjusted we were ready to begin. Mathematicians are almost divisible into two camps according to their script: there are some who write a most beautiful clear hand like Professor Forsyth; and there are also incorrigibly illegible slovens like——, but most mathematicians, and all schoolmasters can unfortunately fill in the blank for themselves. Watson belonged instinctively to the former class, though his fingers were perhaps less resolute towards perfection than his spirit. The particular bit of bookwork was neatly headed, and written out, with negligible erasures or corrections, till the end was reached, when the frown of mild anxiety was replaced by a gratified smile of success with occasionally a statement of the number of years

* Other text-books were beginning to appear. R. S. Heath, a rather senior contemporary of mine, had somehow got hold of Salmon's *Conic Sections*, and knew it by heart. But I doubt whether he ever told Watson.

since there had been any call for that enterprise. And here perhaps I may allow my later experience to suggest what I never thought of at the time—that it must have been pretty difficult to find time and attention for these extra calls on a busy man. Life was strenuous in those days under Percival, who managed to get the last ounce out of men and boys. Besides organising the teaching (and did more examining fall on the staff then than now ?), and taking his full share in it, Watson had charge of a " small " house, occasional sermons to preach in chapel, and a largish family of his own. I am not exalting his cares above those of many others who have had equal or greater : I only wish to assure them all that we boys, who were happily oblivious at the time what they were doing for us, do come in later years to remember and reflect upon it with deep gratitude. Individual methods of course vary. To some we owe an enthusiasm—the outcome, it may be, of a single chance remark. That was scarcely Watson's way. He represented neatness and the orderliness of the beaten track. The mathematics of those days both gained and suffered in this way, and since that time the sufferings have led to the successful exploration of byways ; but even these successes are apt to remind us occasionally of the restfulness of the main road. Watson seemed to inspire order : I never remember his having any classroom difficulties. He wrote out as neatly on the blackboard as on paper, after substituting for the scraping of the pen the preliminary of wrapping the finger-end of his chalk in a piece of paper. How he came to join so rebellious an Association as ours I do not know (would it have made a difference, I wonder, if it had been proposed to " reform " geometrical teaching instead of " improving it " ?), one may infer that the list of names must have been above criticism, and he found an irreproachable part to play as treasurer.

I hope I have not given the impression of a hard man, in mentioning this addiction to orthodoxy : it would be a fatally wrong impression. Kindliness was one of Watson's strongest features, as I had good cause to know. He had a happy way of forgetting anything disagreeable that befell him or others : if by chance reminded of it by some one else overtaken by a fit of contrition, he could scarcely be persuaded that it had ever happened. One of his colleagues has told me that in giving out work to the mathematical staff he would let everyone else choose first, himself accepting as his share what was left. Small wonder that such colleagues can borrow a leaf from his book by declaring that they have forgotten the rigidity of his system and remember only the kindness that made it endurable. He was a good French scholar (I fancy he had French relatives), and would gather a few of us to read a play of Molière in his study : and there were the pleasantest of suppers occasionally on Sunday nights. That was in the days when he had a " small " house. Later he started a big one, which will keep his name permanently as the first administrator. They say it did not distinguish itself much in the athletic world : that is quite possible, for one does not associate Watson's methodical ways with the strenuous excesses of boys' games. But I am quite sure that the house was well ordered, and that the boys in it had a mother's care from Mrs. Watson.

OXFORD, 1925, *June* 1. H. H. T.

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323. Mr. Bowles remarks that this mound (in the garden of the Castle Inn at Marlborough), Silbury Hill, and the mound at Marden form a triangle, which of necessity they must, unless they lie in a straight line.—Art. " Avebury," *Penny Cyclopaedia.*

324. (Buffon's) translations of Hales's *Vegetable Statics*, and of Newton's *Fluxions*, both of which he prefaced with great ability, appear to have been undertaken with a view of improving his style as well as of advancing his knowledge.—" Buffon," *Penny Cyclopaedia.*

MATHEMATICAL NOTES.

777. [V. 1.] A printer sometimes deals with a question so as to suggest a supplementary question.

A question recently set at Charterhouse took this shape in proof.

> "A man rows up stream and passes a floating log 2 miles from his starting point . . . "

This suggests the further question, "How many miles from his starting point must the teacher go before log 2 floats?" C. O. TUCKEY.

778. [K¹. 1.] *Note on Note* 759, *Gazette*, xii. p. 342.

The neat solution by Mr. Travers easily suggests an extension to the more general theorem of Apollonius (De Sectione Spatii).

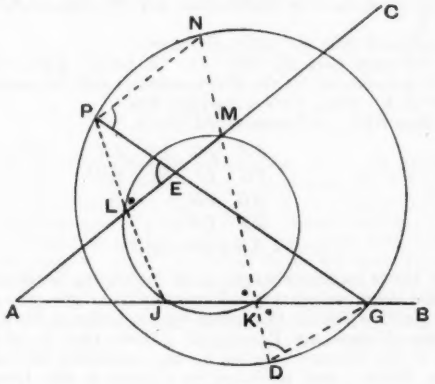

FIG. 1.

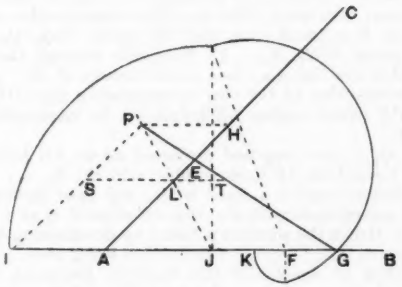

FIG. 2.

K, *L* are two given points on two fixed straight lines *AB*, *AC*; *P* is a given pt.; it is reqd. to draw a st. line *PEG* cutting *AB*, *AC* in *G*, *E* such that *KG*. *LE* is equal to a given rectangle.

Join *PL*, and let it cut *AB* in *J*.

Through J, K, L describe a $\odot$ cutting AC in M.

From P draw $PN \parallel AC$ meeting KM in N, and on MK take KD such that $PL \cdot KD$ = given rect.

Through P, D, N describe a $\odot$ cutting AB in G.

Join PG cutting AC in E.

PEG is the reqd. line.

For $$G\hat{K}D = J\hat{K}M = P\hat{L}E,$$

and $$K\hat{D}G = N\hat{P}G = P\hat{E}L;$$

$$\therefore\ KG : KD = PL : LE;$$

$$\therefore\ KG \cdot LE = PL \cdot KD = \text{given rect.}$$

A different solution of the restricted problem in the L.U.K.* treatise on *Geometry* (by Pierce Morton (?)) leads to the following solution of the more general one:

With A, B, C, P, K, L, J as before, draw PH, PI parl. to AB, AC meeting AC, AB in H, I.

Take F on AB such that $KF \cdot LH$ = given rect.

Find G on AB such that $JG \cdot FG = JI \cdot FK$ ($= IJ \cdot KF$). (Carlyle's construction for G is indicated by the two quadrants and the semicircle.) Join PG cutting AC in E. Then PEG is the reqd. line.

Through L draw $SLT \parallel AB$ meeting PI, PG in S, T.

$$\frac{KF}{FG} = \frac{JG}{IJ} = \frac{LT}{SL} = \frac{LE}{EH};$$

$$\therefore\ \frac{KG}{KF} = \frac{LH}{LE};$$

$$\therefore\ KG \cdot LE = KF \cdot LH.$$

Each of the above solutions serves as an interesting illustration of one of the fundamental theorems relating to homographic ranges—the generalisation of the L.U.K. construction for that given by Mr. Lodge in his paper "Homographic Ranges—Elementary Principles" (*Math. Gaz.* v. 83-88), and the generalisation of Mr. Travers' solution for the correlative theorem discovered by Apollonius, 250 B.C., and published by Chasles in his *Aperçu Historique* under the impression that it was an original discovery.

If a series of pairs of points Q, R are taken on AB, AC such that the rect. $KQ \cdot LR$ is constant, it is known that they form homographic ranges. Hence:

In Fig. 1 a pt. D is found such that the pencil $D(Q_1, Q_2, \dots)$ is directly similar to the pencil $P(R_1, R_2, \dots)$. The conic through the intersection of corresponding rays will therefore be a circle through P, D. A third pt. N is found by the intersection of the two corresponding rays DK (which makes KQ zero) and PN (which makes LR infinite). An intersection of this circle with AB gives G.

In Fig. 2 $Q_1, Q_2, \dots$ are supposed projected on to AB from P, so that a second range is formed on AB homographic with $R_1, R_2, \dots$.

QR, which always touches a fixed conic, will pass through P when the projection of Q corresponds with R; that is, when it is at a double point of the two ranges. Hence the solution is found by determining a double point G of the two ranges on AB, Carlyle's construction being used.

Another problem of Apollonius (De Sectione Rationis), in which, with AB, AC, K, L, P given as before, the *ratio* $KG : LE$ is given instead of the rectangle $KG \cdot LE$, has two analogous solutions, the diagrams for which are combined in Fig. 3.

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* The Library of Universal Knowledge.

If a series of pairs of points Q, R is taken along AB, AC such that the ratio $KQ : LR$ is constant, it is known that they form homographic ranges.

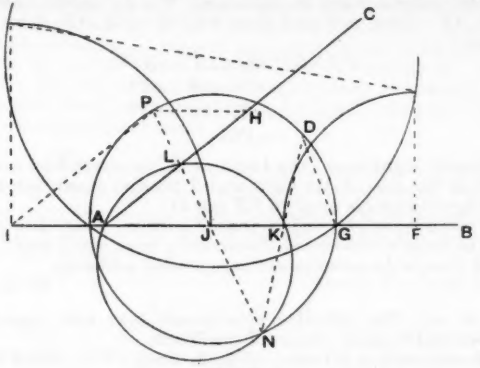


FIG. 3.

As before we may :

either Find the pt. D such that the pencil $D(Q_1, Q_2, \dots)$ is directly similar to the pencil $P(R_1, R_2, \dots)$, so that corresponding rays DQ, PR intersect on a circle through P, D, and find a third point N on this circle by the intersection of the two corresponding rays PL, DK, which make KQ, LR both zero, and obtain the point G by the intersection of this circle with AB;

or Suppose the range $Q_1, Q_2, \dots$ projected on to AB from P, and then find G as the double point of the two homographic ranges on AB. For this, I, F being the points one on each corresponding to that at infinity on the other, J and K being a pair of corresponding points, we have $IG \cdot FG = IJ \cdot FK$. The Carlyle construction for G is indicated by the two quadrants and the semicircle.

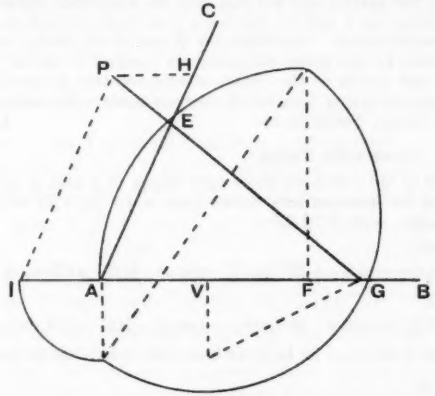


FIG. 4.

The L.U.K. construction for G in the restricted case of the first problem, where K and L fall on A (Fig. 4), readily yields another, given in T. Simpson's

Geometry, Bland's *Geometrical Problems*, Cresswell's *Supplement to Euclid*, and Wright's edition of Hirsch's *Geometry*—though in these works it is obtained by a different analysis based on dissection. For AG having been found by taking $AF.AH$ = given rect. and then $AG.GF = PH.AF$, if we bisect AF in V, we have

$$\begin{aligned} IV^2 - IA^2 &= 2IA.AV + AV^2 \\ &= IA.AF + AV^2 \\ &= AG.FG + AV^2 \\ &= VG^2 ; \end{aligned}$$

and it follows G might have been found as in the above four works, *i.e.* by getting VG as the base of the right-angled triangle shown below it, whose height and hypotenuse are equal to PH and IV.

There is much ingenious work connected with the above two problems of Apollonius in Swale's *Geometrical Amusements*, from which useful hints may be obtained though the arrangement is somewhat confusing.

E. M. LANGLEY.

779. [T. 2. b.] The following approximate laws with regard to ropes may be of interest to some readers of the *Gazette*.

If the circumference is C inches, 10 yards weigh C^2 lb., and if it is a good manilla rope it will bear 1000 C^2 lb. tension. This is somewhat excessive for very thick ropes, but for moderate sizes is very accurate. Thus for $C=3$, it makes 10 yards weigh 9 lb., and max. tension 9000 lb. A trade book from which I got my data makes weight $=8\frac{3}{4}$ lb., tension 9000 lb. Pretty good, especially the tension!

A rougher formula, obtained by taking $\pi^2 = 10$, is D^2 = weight of one yard, where D is the diameter. But this is about $1\frac{1}{2}$ per cent. higher than the C^2 formula.

A. LODGE.

780. [L¹. 15. b.] *Note on Reciprocation with respect to a Circle.*

In the text-books to hand, which of the angles between two lines is equal to the angle subtended by the corresponding points is not fully identified, and the slurring of the matter has led not only to inaccurate statement in some envelope problems (as I believe has been previously remarked on), but also to error in metrical work. The difficulty is one of emphasis, and can be got over by additions to the usual statement as marked in italics. "The angle subtended by two points at the centre of reciprocation is equal to *that* angle between the corresponding lines *which does not contain the centre.*

University College, Nottingham. H. G. GREEN.

781. [X. 2.] *Home-made Tables.*

If the values of $\sin x$ and $\cos x$ for two values of α and β of x are known, approximations for intermediate values from α to $\frac{1}{2}(\alpha+\beta)$ are given by the following formulae, with $0 \leqslant h \leqslant \frac{1}{2}$:

$$\begin{aligned} &\sin\{\alpha + h(\beta-\alpha)\} \\ &\quad = \sin\alpha + h(\beta-\alpha)\cos\alpha + 2h^2\{(\sin\beta - \sin\alpha) - \tfrac{1}{4}(\beta-\alpha)(3\cos\alpha + \cos\beta)\}, \\ &\cos\{\alpha + h(\beta-\alpha)\} \\ &\quad = \cos\alpha - h(\beta-\alpha)\sin\alpha - 2h^2\{(\cos\alpha - \cos\beta) - \tfrac{1}{4}(\beta-\alpha)(3\sin\alpha + \sin\beta)\}. \end{aligned}$$

For the range from $\frac{1}{2}(\alpha+\beta)$ to β we have the equivalent formulae:

$$\begin{aligned} &\sin\{\beta - k(\beta-\alpha)\} \\ &\quad = \sin\beta - k(\beta-\alpha)\cos\beta - 2k^2\{(\sin\beta - \sin\alpha) + \tfrac{1}{4}(\beta-\alpha)(\cos\alpha + 3\cos\beta)\}, \\ &\cos\{\beta - k(\beta-\alpha)\} \\ &\quad = \cos\beta + k(\beta-\alpha)\sin\beta + 2k^2\{(\cos\alpha - \cos\beta) + \tfrac{1}{4}(\beta-\alpha)(\sin\alpha + 3\sin\beta)\}. \end{aligned}$$

In these formulae, $\beta - \alpha$ must be expressed in radians on the right, but otherwise the angles may be taken directly in whatever units are convenient.

With $\beta - \alpha$ as large as 15°, these formulae give intermediate values not more than one unit wrong in the fifth place of decimals. In other words, we can determine values accurate to that order for any angle whatever from the values for 30° and 45° alone, after filling in the value for 45° − 30°.

To a reader who objects that he would rather carry five-figure tables in his pocket than such formulae as the above in his head, I can only offer to point out how the formulae are obtained. E. H. N.

782. [K¹. 6. a.] *The Equation of a Line perpendicular to a Given Line.*

Given line, $U \equiv Ax + By + C = 0$; intercepts $-C/A$, $-C/B$. Draw the $\perp^r$ U', from the point, Q, at which U cuts the y-axis; and let the intercepts of U' be X and Y. A graph shows that X has the opposite sign to $-C/A$, while $Y = -C/B$; and by Euc. VI. 8,

$$-X : Y, \text{ or } -C/B :: -C/B : -C/A \text{ [whence } X = AC/B^2\text{]},$$

or

$$1/X : 1/Y :: B : -A.$$

Hence $U' \equiv Bx - Ay + C' = 0$ [where $C' = -AC/B$, but this is not required for the argument]. The equation of the $\perp^r$ from any other point $P(h, k)$ will, be the same except for C', say $U'' \equiv Bx - Ay + C'' = 0$ [$\because$ the two $\perp^{rs}$ are parallel, and $\therefore$ the ratios of the intercepts the same]; and since $P(h, k)$ is on U'', $Bh - Ak + C'' = 0$, or $C'' = -Bh + Ak$. Hence

$$U'' = Bx - Ay - (Bh - Ak) = 0,$$

or

$$B(x - h) - A(y - k) = 0.$$

G. J. LIDSTONE.

783. [V. 1. a. μ.] *Forces in Equilibrium along the Sides of a Plane Quadrilateral.*

By considering forces in equilibrium along the sides of a plane quadrilateral, Mr. R. F. Davis has shewn * that if the quadrilateral is circumscribed to a circle, then the following condition is satisfied:

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A,$$

$ABCD$ being the quadrilateral, a being the length of the side AB, etc.

Mr. Davis writes: "and I think this should come out equivalent to

$$a + c = b + d."$$

It is of some interest to notice that the equivalence is a corollary from the equation

$$\sin A \sin B \sin C \sin D[(a+c)^2 - (b+d)^2]$$
$$= (a \sin A \sin B + c \sin C \sin D)^2 - (b \sin B \sin C + d \sin D \sin A)^2,$$

which holds for any quadrilateral, and may be deduced from the formulae:

$$\text{Area } ABCD = \frac{a^2 \sin A \sin B - c^2 \sin C \sin D}{2 \sin (A+B)} = \frac{b^2 \sin B \sin C - d^2 \sin A \sin D}{2 \sin (B+C)}$$

and

$$\frac{\sin A \sin D - \sin B \sin C}{\sin A \sin B - \sin C \sin D} = \frac{\cos (A-D) - \cos (B-C)}{\cos (A-B) - \cos (D-C)} = -\frac{\sin (A+B)}{\sin (B+C)}.$$

There are advantages, however, in dealing with Mr. Davis's problem without reference to the angles of the quadrilateral.

If the sides BA, CD intersect in a point which divides them externally in the ratios $\lambda : \lambda + 1$ and $\mu : \mu + 1$ respectively, then it can be shewn that forces along AB, CB, CD and AD to be in equilibrium must be proportional to $\lambda(\lambda+1)a$, $\lambda\mu b$, $\mu(\mu+1)c$ and $(\lambda+1)(\mu+1)d$ respectively.

* *Math. Gazette*, 12 (December 1924), 255. Note 746.

λ and μ are related by the equation

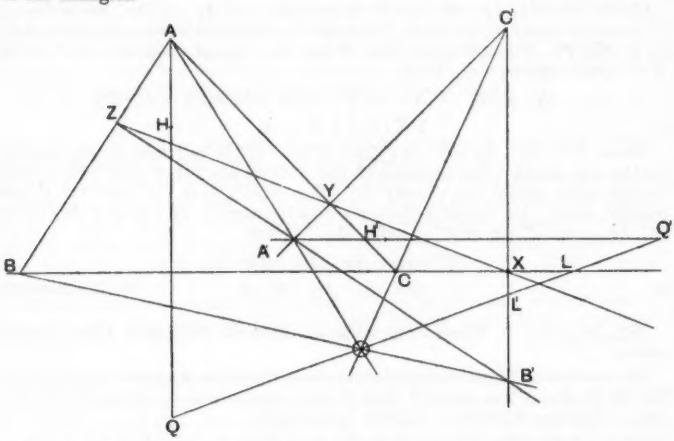
$$(\mu-\lambda).[\lambda(\lambda+1)a^2-\mu(\mu+1)c^2]=\lambda\mu b^2-(\lambda+1)(\mu+1)d^2,$$

and it follows that the condition for the sides of the quadrilateral to touch a circle may be written as

$$\lambda(\lambda+1)a+\mu(\mu+1)c=\lambda\mu b+(\lambda+1)(\mu+1)d,$$

or as $$a+c=b+d.$$ F. J. W. W.

784. [K¹. 2. a.] If from the points of intersection of a transversal with the sides of a triangle, perpendiculars be drawn to the sides, forming a new triangle, to prove that the transversal bisects the line joining the orthocentres of the triangles.



ABC is one triangle, XYZ the transversal, $A'B'C'$ the other triangle (directly similar to ABC) and O is the centre of perspective. This point is also one of the points of intersection of the circumcircles of the triangles, for $A'\hat{A}Y=A'\hat{Z}Y$ and $B'\hat{Z}X=B'\hat{B}X$; $\therefore\ O\hat{A}C=O\hat{B}C$, and similarly for O, A', B', C'.

Let the perpendicular from A to BC meet the circle ABC in Q, and let QO meet the perpendicular from A' on $B'C'$ in Q'. Then

$$B'\hat{O}Q'=B\hat{O}Q=B\hat{A}Q=B'\hat{A}'Q',$$

so that Q' is on the circle $A'B'C'$. Let QOQ' meet BC in L and $B'C'$ in L'. Then $X\hat{L}L'=A'\hat{Q}'O=A'\hat{B}'O\equiv Z\hat{B}'B=Z\hat{X}B$. Hence XYZ bisects LL'.

Now the Simson's Lines of O with regard to the circles ABC and $A'B'C'$ are both parallel to XYZ, and hence HL, $H'L'$ are both parallel to XYZ, where H, H' are the orthocentres of the triangles ABC, $A'B'C'$.

Hence XYZ bisects HH'. N. M. GIBBINS.

785. [J. 2. c.] *Solution of Problem, Gazette*, xii. p. 339, *Note* 752.

Minoru and St. Amant	-	·375 138 000 588 602 815 283 2 each.
Game Chick	- - -	·188 429 586 502 664 794 931 2 ,,
Miss M'Giggle	- - -	·042 363 203 265 638 904 627 2 ,,
Gou-Gou	- - - -	·018 931 209 054 490 669 875 2 ,,

W. HOPE-JONES.

786. [V. a. 1.] *Tracing the Conic.*

In certain methods of drawing a conic given by an equation of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$, there is left an ambiguity as to the position of the conic with reference to its axes. The following are simple analytical tests which determine its position, and can be used as checks in methods where no similar ambiguity arises.

Parabola. $h^2=ab$, or $C=0$.

The equation may be written

$$(ax+hy)^2+2agx+2afy+ac=0,$$

or
$$(ax+hy+\lambda)^2=2(a\lambda-ag)x+2(h\lambda-af)y+ac-\lambda^2.$$

The value of λ giving the axis and the tangent at the vertex is given by

$$\lambda(a^2+h^2)=afh+a^2g.$$

Substituting in this value, and using the fact that $0=aG+hF+C=aG+hF$, we obtain for the equation of the parabola

$$(ax+hy+\lambda)^2=\{a^2/(a^2+h^2)\}(Gx+Fy)+ac-\lambda^2.$$

This, referred to parallel axes through the vertex, gives the equation

$$(ax+hy)^2=\{a^2/(a^2+h^2)\}(Gx+Fy).$$

The parabola cuts the latter axes at the origin and at $(G/(a^2+h^2), 0)$ and $(0, Fh^2/a^2(a^2+h^2))$.

Hence, calling the "axis of the parabola" that part of the axis "inside" the parabola, the axis of the parabola is in the quadrant (of the new coordinate axis) in which x is + or − and y is + or − according as G is + or − and F is + or − respectively. In numerical work for G, F read the coefficients of x, y on the right-hand side (which are positive multiples of G, F respectively).

Ellipse. Suppose the ellipse has been referred to parallel axes through the centre, and its equation is

$$ax^2+2hxy+by^2=1.$$

The coordinate axes may be turned through a positive acute angle θ (if $h\neq 0$) to coincide with the principal axes. We then have $\sin 2\theta>0$.

If (X, Y) are the coordinates of (x, y) referred to the new axes,

$$x=X\cos\theta-Y\sin\theta,\quad y=X\sin\theta+Y\cos\theta.$$

The equation of the conic reduces to

$$a'X^2+b'Y^2=1,$$

if
$$2h\cos 2\theta+(b-a)\sin 2\theta=0,$$

i.e.
$$\tan 2\theta=2h/(a-b)\left(0<\theta<\frac{\pi}{2}\right);$$

and
$$a'=a\cos^2\theta+b\sin^2\theta+h\sin 2\theta,$$
$$b'=a\sin^2\theta+b\cos^2\theta-h\sin 2\theta.$$

Thus
$$\begin{aligned}a'-b'&=(a-b)\cos 2\theta+2h\sin 2\theta\\&=2h\sin 2\theta\{1+\cot 2\theta\,(a-b)/(2h)\}\\&=2h\sin 2\theta\{1+(a-b)^2/(4h^2)\}\\&=\sin 2\theta/(2h)\{(a-b)^2+4h^2\}.\end{aligned}$$

This is positive or negative according as $h>$ or <0. Hence the longer or shorter axis is in the positive quadrant (of the old axes) according as $h<$ or >0.

Hyperbola. The calculation is the same as for the ellipse. If $a'>b'$ and one of a', b' is negative, then $a'>0$ and $b'<0$. Hence the real or imaginary axis is in the positive quadrant according as $h>$ or <0. H. V. MALLISON.

787. [V. 1. a. μ.] J. D. Everett, *Units and Physical Constants* (1879). Preface: "A still greater innovation has been introduced" (since the previous edition, 1875) "in an extended use of the symbols and processes of multiplication and division, in connection with equations which express not numerical but physical equality. The advantages of this mode of procedure are illustrated by its application to the solution of the most difficult problems on units that I have been able to collect from standard text-books."

$$\frac{\text{yard}}{\text{minute}^2}=\frac{1}{1200}\,\frac{\text{foot}}{\text{second}^2}. \quad \text{(P. 8.)}$$

$$t^2=(\text{min.})^2\,\frac{\text{inch}}{50\text{ ft.}}=\frac{(\text{min.})^2}{600}=6(\text{sec.})^2. \quad t=\sqrt{6}\text{ sec.} \quad \text{(P. 10.)}$$

And forty-three years afterwards the "Stroud system" was being put forward at the B.A. as unknown. E. H. N.

788. [V. 8.] *Scottish Historical Note.—MacLaurin, Stirling, and Campbell.* In the correspondence that passed between MacLaurin and Stirling, which is published in the *Life and Letters of J. Stirling*, is found a reference to a battle of Tracts between MacLaurin and Geo. Campbell regarding certain algebraic theorems, printed in the *Philosophical Transactions* of 1729. I have found that the titles of these Tracts are as follows:

(1) Geo. Campbell.
Remarks on a paper published by Mr. MacLaurin in the *Philosophical Transactions* for the month of May, 1729.
(No date or name is given.)

(2) MacLaurin.
A Defence of the Letter published in the *Philosophical Transactions* for March and April, 1729, concerning the impossible Roots of Equations, in a Letter from the author to a friend at London.
Edinburgh, 1730.

Campbell's Tract is in the Advocates' Library, Edinburgh. No other copy of No. (1) is known to exist, and I have just presented my own copy of No. (2) to the Advocates' Library. Another copy of No. (2) is in Edinburgh University Library, and is the only other copy known.

Reference is frequently made in biographical notices of James Stirling to a MS. treatise in two volumes on *Weights and Measures*. I have recently seen these volumes, and regret to say that this reference is not correct. The two books are obviously in Stirling's handwriting, but the one volume is simply a revised copy of the other, and contains only a short historical introduction to the subject, which is probably of little value. From internal evidence we learn that it was written after the death of his friend Ffolkes.
C. TWEEDIE.

789. [A. 5. b.] *An Early Work on Interpolation.*

I notice in a review (*Math. Gaz.* xii. 220) of Whittaker and Robinson's *Short Course on Interpolation* it is stated that "the subject has not yet found a text-book home." It may interest the Reviewer to know that I have in my possession a text-book entitled *Interpolation, Summation and Adjustment of Numerical Tables*, by W. S. B. Woolhouse, published in 1865 by C. and E. Layton, 150 Fleet St. It seems to be a very exhaustive and useful publication for its day. G. T. GILLESPY.

790. [V. 1.] *A Simple Piece of Apparatus.*

Take a cylindrical cigarette tin with lid on. Bore holes in centres of lid and bottom, sufficiently large to take a hat-pin about which the box is to be

rotated. Cut out three pieces of plain paper, A, B and C. A and B are circular sheets the size of the lid; C is rectangular, with breadth equal to height of box and length equal to circumference of lid.

On A mark a point P, not the centre; on B draw a chord, not the diameter. Paste A and B on lid and bottom of box, outside. When box is rotated with axis horizontal, point P describes a circle, and actually appears as such if the box is rotated quickly enough. A thus serves to illustrate the locus of a moving point which is always equidistant from a fixed point.

Pupils find some difficulty in grasping the subject of "Loci" in the early stages, and it will be found that this simple experiment helps towards clarifying their conception of a locus.

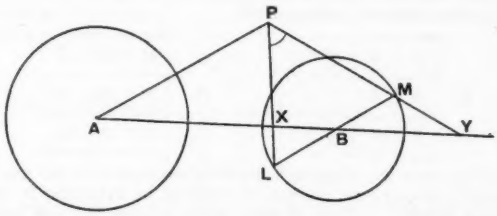
Similarly, B is used to shew that a moving straight line under certain conditions envelopes a circle.

With regard to C, this may be used in connection with the introductory work on the area of irregular figures. Taking one long edge of C as axis, draw ordinates, equidistant and parallel, adjacent ordinates always differing in length by the same small amount and being connected by a line parallel to the axis through the end of one ordinate, so that a series of rectangles is formed. Paste C round the curved side of the box, and rotate the box rapidly with the axis of rotation vertical. This has the effect of (a) apparently reducing the distance between the ordinates and, at the same time, (b) making the series of steps at the top ends of the rectangles appear as a continuous line. This association of indefinitely thin rectangular strips with the coincidence of a staircase-like and a continuous line boundary is what we want pupils to realise clearly in order to understand thoroughly the application of the method of limits to areas. The apparatus helps in this direction.

R. S. WILLIAMSON.

791. [K¹. 11. b.] *The Circle of Similitude of Two Circles is Coaxal with them.*

The following "proof" is a curiosity, though it would probably meet with the disapproval of examiners.



When the centres of similitude of the circles are real, their points of intersection are imaginary. Do not be deterred by this, but let P be one of these imaginary points, and join P to the centre A of the first circle. In the second circle let LBM be a diameter (imaginary) parallel to AP. Then, by a well-known theorem, PL, PM pass through the centres of similitude X and Y.

Now P is a point on the second circle: hence $L\hat{P}M$ is a rt. $\angle$ (imaginary). The circle on XY as diameter thus passes through P, and, since it will likewise pass through the other point of intersection, it is coaxal with the original circles.

The method is, of course, quite sound if all its assumptions are first justified.

R. C. J. HOWLAND.

792. [C. 2. h.] *Integration of x^n by Direct Summation.*

A general method of summation for $\sum_{p=1}^{p=k} p^n$ is to write

$$p^n = Ap(p+1)\dots(p+n-1) + Bp(p+1)\dots(p+n-2) + \dots + Rp(p+1) + Sp + R.$$

The coefficients are most readily found as the remainders in successive divisions by p, $p+1, \dots, p+n-1$. All, however, that concerns us here is that A is evidently unity. From a general formula, we then have

$$\sum_{p=1}^{p=k} p^n = \frac{k(k+1)\dots(k+n)}{n+1} + B\frac{k(k+1)\dots(k+n-1)}{n} + \text{etc.}$$
$$= \frac{k^{n+1}}{n+1} + \text{smaller powers of } k.$$

Hence
$$\int_0^m x^n dx = \lim_{k\to\infty} \sum_{p=1}^{p=k} \left(\frac{pm}{k}\right)^n \frac{m}{k}$$
$$= \lim_{k\to\infty} \left(\frac{m}{k}\right)^{n+1} \sum_{p=1}^{p=k} p^n$$
$$= \lim_{k\to\infty} \left(\frac{m}{k}\right)^{n+1} \left(\frac{k^{n+1}}{n+1} + \text{smaller powers of } k\right)$$
$$= \frac{m^{n+1}}{n+1}.$$

R. C. J. HOWLAND.

793. [X. 8. a.] *A New Type of Magic Square.*

In the usual type of magic square, the integers forming any row, column or diagonal have a sum which is the same for all such sets. The present problem is to find squares of numbers in which the *product* of the integers in any row, column or diagonal is the same, and to find a means of constructing all such squares.

It is at once obvious that squares of this kind may be derived from the old form as follows. Choose any integer a, and replace each number n in the old type of square by a^n. Then the resulting square is of the required type. Thus, starting with the square I. and using the base 2, we arrive at the square II., in which the constant product is 2^{15}.

I.

4	9	2
3	5	7
8	1	6

II.

16	512	4
8	32	128
256	2	64

Starting now from the other end, consider any square of the new type, and let the product of any row, column or diagonal be expressed in terms of its prime factors $a^p b^q c^r \dots$. Then if from every number in the square its a factors are extracted, these taken alone will form a square of the new type in which the constant product is a^p, while in this square the indices of a would form a square of the old type.

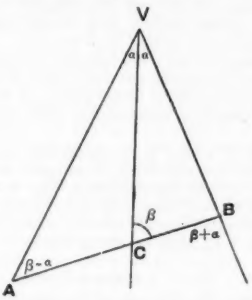
Hence the possible squares of the new type are all obtained by either: (1) transforming a square of the old type by making its numbers into indices of any prime base; or (2) multiplying together the corresponding elements of two squares of equal dimensions formed by the method (1). R. C. J. HOWLAND.

794. [L². 20.] *The volume cut from a right circular cone of semi-vertical angle α by a plane making an angle β ($> \alpha$) with its axis.*

Let VC be the axis of the cone, and let a plane Σ making an angle β ($> \alpha$) with VC cut the cone in an ellipse whose major axis is AB (so that Σ is perpendicular to the plane VAB).

Then the eccentricity of the elliptic section so obtained is $\cos\beta/\cos\alpha$, and its area is

$$\begin{aligned}&\tfrac{1}{4}\pi AB^2 \sec\alpha \,.\, (\cos^2\alpha - \cos^2\beta)^{\frac{1}{2}}\\ &= \tfrac{1}{4}\pi AB^2 \sec\alpha \,.\, \{\sin(\beta-\alpha) \,.\, \sin(\beta+\alpha)\}^{\frac{1}{2}}\\ &= \tfrac{1}{4}\pi AB \,.\, \sec\alpha \,.\, \{VB \,.\, \sin 2\alpha \,.\, VA \,.\, \sin 2\alpha\}^{\frac{1}{2}}\\ &= \tfrac{1}{2}\pi \,.\, AB \,.\, \sin\alpha \,.\, (VA \,.\, VB)^{\frac{1}{2}}.\end{aligned}$$



The perpendicular distance of V from the plane Σ is

$$2\Delta VAB/AB = VA \,.\, VB \,.\, \sin 2\alpha/AB.$$

Hence the volume of the solid cone VAB

$$= \tfrac{1}{3}\pi \,.\, (VA \,.\, VB)^{\frac{3}{2}} \sin^2\alpha \,.\, \cos\alpha.$$

This expression, which is not usually given in text-books on Mensuration, is of a form suitable for practical application.

East London College. S. L. GREEN.

795. [K¹. a.] *The Condition that an Equation of the Second Degree in Trilinear Coordinates should represent two Straight Lines.*

Let $U=0$ be the equation of x, y, z, the coordinates of the centre.

The equation of the polar of (xyz) is

$$X\frac{\partial u}{\partial x} + Y\frac{\partial u}{\partial y} + Z\frac{\partial u}{\partial z} = 0.$$

Since the polar at the centre is the line at infinity,

$$\frac{\frac{\partial u}{\partial x}}{a} = \frac{\frac{\partial u}{\partial y}}{b} = \frac{\frac{\partial u}{\partial z}}{c} = k \text{ (say)}.$$

But if the equation represents two straight lines the centre is on the curve.

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$\therefore\ k(ax+by+cz) = 0.$$

Hence $k=0$, since $ax+by+cz$ is constant and not zero.

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0.$$

Hence the condition is, with the usual notation,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

A. W. LUCY.

REVIEWS.

Mathematical Theory of Life Insurance. By C. H. FORSYTH. Pp. 74 + vii. 6s. 6d. 1924. (Wiley; Chapman, Hall.)

This book is primarily intended for students of colleges in the United States taking mathematical courses in investment and finance. It gives a simple statement of life contingencies, and considering its small compass it should prove useful to those for whom it is written. Generally speaking the explanations are clear, and we may recommend specially the questions "for discussion" of the fallacy of assessment insurance and the chapter on valuation. The latter gives a good short description of the methods used in the States; these methods are not adopted in this country, and are seldom discussed here.

The book shows signs of haste in preparation, as there are many slips: *e.g.* the use of unexplained symbols in examples on p. 16; an imperfect definition of single premiums for assurance, p. 25; a recommendation to read an English text-book now out of print and replaced by another with a different title (p. 65), etc. The English reader should avoid the preface unless he is searching for examples of "English as she is wrote." W. PALIN ELDERTON.

Plans and Elevations. By V. LE NEVE FOSTER. 1s. 6d. (Bell & Sons.)

The growing complexity of the school curriculum, and the consequent difficulty in fitting into the time-table adequate periods for the subjects it is required to include, render welcome any such work as this collection of exercises in Descriptive Geometry calculated to enable a teacher to use the small intervals at his disposal more effectively. It consists of 33 leaves of questions, each with the lines of reference and the lines supposed given already marked in its proper place. The leaves are perforated so as to allow them to be detached and collected. It can be strongly recommended for private as well as for class use.

It would be well in another issue to add to the statement in the preface that the method of Descriptive Geometry was invented by Monge, the further information that a cheap edition of his *Géometrie Descriptive* has been recently published in the series of "Les Maîtres de la Pensée Scientifique" by Gauthier-Villars.

History of Mathematics. Vol. II. (Special Topics of Elementary Mathematics.) By DAVID EUGENE SMITH. Pp. xii + 725. 21s. net. 1925. (Ginn & Co.)

We congratulate Dr. Smith on the completion of his onerous labours of research and selection, and the teaching world on the valuable addition made to its resources by the publication of the results. He has certainly succeeded well in his object—that of "supplying teachers and students with a usable text-book on the history of elementary mathematics," the term "elementary" expressly including the first steps of the calculus, and actually extended to determinants, geodesy, and non-euclidean geometry. Vol. I., reviewed in No. 170 of the *Gazette*, treated the subject in periods, with constant attention to geographical and historical backgrounds, assessing the contributions of various races of mankind. Vol. II., as indicated by its title, treats the subject by topics, so that the reader can study without interruption a continuous account of one subject, such as Arithmetic or Algebra. The following brief outline of the 10 pages devoted to Analytical Geometry may serve as an example:—(i) The idea of coordinates seems to have occurred to Egyptian surveyors. Greek geographers and astronomers treated points by longitude and latitude. Roman towns were laid out with respect to two rectangular axes, the *decimanus* and the *cardo*. Greek geometers in their treatment of geometrical figures made use of what are substantially two rectangular axes, Apollonius giving what is merely a rhetorical statement that $x = ay - b$ represents a straight line, though, as Sir T. Heath points out, the Greeks aimed rather at expressing equations between areas as shortly as possible rather

than at making the fixed lines of the figure as few as possible. In the Middle Ages, Oresme made a distinct advance with his longitudines and latitudines, equivalent to our abscissae and ordinates, the method becoming the subject of university lectures as early as 1398.

(ii) Though Greek writers used Euclid's II. 4 for extracting square roots, it is among Arab and Persian writers that we first find geometric figures in works devoted solely to Algebra, Omar Khayyam recognising the one-to-one correspondence between Algebra and Geometry more than his predecessors. Among Europeans, Fibonacci was the first to recognise the value of this correspondence, Regiomontanus, Pacioli and Cardan also availing themselves of the relation, the recognition of which became subsequently common.

(iii) Though Geraldi (1630) brought out a text-book on algebraical Geometry, probably Harriot, Fermat and Descartes may be regarded as the originators of the Analytical Geometry of to-day, Descartes being the first to publish (1637) a treatise on it, *La Géométrie*. After the works of De Witt (1658) and Lahire (1679) the elements may be regarded as having become established, the most remarkable single contributor being Newton.

Of modern contributors Plücker stands easily foremost, modern abridged notation and the present analytic form of the principle being given in his *Analytisch-geometrische Entwickelungen* (1828-1831).

The sketch outlined above is followed by a list of about 30 special curves with their equations and historical notes.

Like its predecessor, the new volume (i) is illustrated profusely by reproductions of MSS. and of diagrams, prints and text from old printed books, well calculated to arouse and stimulate interest; *e.g.* pages from the *Crafte of Nombrynge* and Cardan's *Ars Magna*; representations of the various forms of Abacus and their use; diagrams for early Japanese attempts at integration; the picture of the 1597 Range-finder; and (ii) devotes considerable space to Oriental nations; (iii) gives copious references throughout to original sources; (iv) supplies at the end of each chapter a page of "topics for discussion," rather with the object of encouraging the student to pursue his reading in other works than to examine him in what he has just read.

Of one subject—Perspective—we should have welcomed an extended sketch. Dr. Smith's extends little beyond the ages during which the term was almost confined to what is now called Geometrical Optics, and contains no reference to the advances in the theory made by Del Monte, Stevinus and s'Gravesande. Brook Taylor's books on Linear Perspective were indeed mentioned in Vol. I., but without anything beyond the mention of his use of vanishing points to show the reader that the term "perspective" is there used in a different sense from that in the earlier sections of the book. The intimate connexion of Brook Taylor's method with the work of Desargues and with the plane transformation given by Newton in Lemma XXII., in fact its development as the application of Central Projection, might with advantage have been pointed out.

Hints on Notemaking in Science and Mathematics. By R. T. HUGHES. Pp. 132. 2s. 6d. 1925. (Bell & Sons.)

A compact little treatise, showing throughout marks of the experience of a practised teacher. It deals thoroughly with its subject from the various points of view from which it may be regarded as the ages and conditions of the notemaker are taken into account. It has also shrewd and sound remarks on kindred matters, which, though of great importance in teaching, are apt to be neglected; *e.g.* the need that not only the form-master or "English" master, but also that the "Science" master (we would say that *every* master) should *always* insist on good English in the answers of his pupils. We quite agree with the writer that these are inclined to think that "English does not matter in Science." From our experience, chiefly in mathematical sets, we are inclined to think that their point of view was more general, viz. that English did not matter in any subject in which the teacher was weak enough to accept slipshod expressions. It is true that "the pressure of many other things often makes him remiss in this matter," but such remissness brings its own punishment, and enlightened self-interest, to say nothing of

higher motives, should ensure constant attention to this duty. Mr. Hughes, whether writing from his own experience or not, will probably recall that of other careful teachers when he says that "notes, unfortunately, are too often judged by others simply by their appearance; little or no consideration is paid to the conditions under which they should have been written and to the contents. . . . *Schoolwork is often inspected by people with no practical knowledge of scientific method, who are greatly impressed by neatness, by coloured inks, and by beautiful diagrams.*" The book, though specially intended for students and teachers of science, may be profitably studied by all interested in sound education. Those whose special subject is Mathematics have a section at the end, in which their duties as individuals and as loyal colleagues are discussed. Among the general considerations, however, in Chapter I. are some of great and increasing importance to the teacher of Mathematics—those relating to Drawing. The representation of figures in space is demanding attention in his own subject as well as in the subjects taught by his science colleagues. (See *The Teaching of Geometry in Schools*, pp. 22-25.) It is by no means the least advantage of the simplification of courses in Plane Geometry that the Elements of Solid may be taken much earlier than they used to be. If effective use is to be made of this possibility, some time must be devoted to drawing and representations of solid figures and enabling the pupil to recognise the different planes of their bounding surfaces and possible sections *without shading*. In some text-books effective use is made of this device, but it is unsuitable for pupils, who should not be allowed to attempt it. He should learn to draw models and pieces of apparatus of gradually increasing complexity. When the time comes for the formal study of Solid, more systematic instruction in such representation must be given. It is to be hoped that sections on Plan and Elevation, on Perspective and on various projective devices used by craftsmen, will soon be looked upon as necessary adjuncts to any text-book on Solid Geometry. E. M. LANGLEY.

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G. VERRIEST Cours de Mathématiques Générales; II - 1925

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Plans and Elevations. By V. LE NEVE FOSTER. Pp. 34. 1s. 6d. 1925. (Bell & Sons.)

Engineering Mathematics. By R. W. M. GIBBS. Pp. xi+190+xii. 4s. 1924. (Blackie.)

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The Baconian Method of Scientific Research. By F. CAJORI. Reprint from *Science Monthly.* Jan. 1925. Pp. 85-91.

Notes on the Interpretation of Correlation Measures. By E. R. HAMILTON. Pp. 7. Reprint from the *Forum of Education.* Nov. 1924.

The Judgment of Quality in Handwriting. By E. R. HAMILTON. Pp. 4. Reprint from *Educational Research.* Dec. 1924.

The Ideal Aim of Physical Science. By E. W. HOBSON. Pp. 34. 2s. net. 1925. (Cambridge University Press.)

Calculus of Variations. By G. A. BLISS. Pp. xiii+189. 2$. 1925. (Open Court.)

Mathematical Theory of Life Insurance. By C. H. FORSYTH. Pp. vii+74. 6s. 6d. net. (Wiley; Chapman, Hall.)

Miscellaneous Exercises in School Mathematics. Compiled by H. E. PIGGOTT and D. F. FERGUSON. Pp. vii+250+xxxii. Part I., 2s. 6d. Complete, 4s. 6d.

The Principles of Thermodynamics. By G. BIRTWISTLE. Pp. ix+163. 7s. 6d. net. 1925. (Cambridge University Press.)

Curve Sghembe Speciali Algebriche e Trascendenti. Vol. i. *Curve Algebriche.* By G. LORIA. Pp. xi+374. L. 65. 1925. (Zanichelli, Bologna.)

Cours de Mathématiques Générales à l'usage des étudiants en sciences naturelles. Part II. *Géométrie Analytique à trois Dimensions. Calcul Intégral.* By G. VERRIEST. Pp. 388. 38 frs. 1925. (Gauthier-Villars.)

A First Trigonometry. By W. G. BORCHARDT and A. D. PERROTT. Pp. x+114 +xvii+xii. 2s. 6d. 1925. (Bell.)

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Benjamin Peirce. Reminiscences by C. W. ELIOT, A. L. LOWELL, W. E. BYERLY, A. B. CHACE; Biographical Sketch, by R. C. ARCHIBALD; The Writings of Pierce. Pp. 1-30. *The Mathematics of Biology.* Pp. 30-36. O. W. RICHARDS.

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Annals of Mathematics. (Princeton University Press, N.J.)

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Bollettino della Unione Matematica Italiana. (Zanichelli, Bologna.)

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Contribución al Estudio de las Ciencias físicas y Matemáticas. (Univ. Nacional de La Plata.) Dec. 1923. Anuario 1924.

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Klima und Sterblichkeit. Eine statistische Untersuchung, ausgeführt im Auftrage des Verbandes Deutscher Winzer in Aserbeidjan, "Konkordia." Pp. 179-194. PAUL LORENZ.

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The Journal of the Indian Mathematical Society. (Varadachari, Madras.)

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On the Instability of the Pear-shaped Figure of Equilibrium of a Rotating Mass of Homogeneous Liquid. Pp. 1-24. S. R. U. SAVOOR. Notes and Questions:—*Inequalities and Ratios.* Pp. 1-10. I. T. IYENGAR.

Journal of the Mathematical Association of Japan. (I. Mori, Tokyo Higher Normal School for Girls.)

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On the Intersections of the Trisectors of the Angles of a Triangle. Pp. 260-262. F. MORLEY.

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